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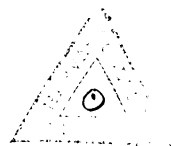
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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

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4329. (W. B. Davis, B.A.)—The highest power of any number published is Mr. Shanks' power of 2 to the 721st power. How may this be verified? Answer—By dividing it by 127, you will get a remainder 1, if you copy it correctly; which may be easily remembered. Objection—This only implies a "*juste rapport*" of 3 in the last place, and a do. for 2 is by no means uncommon. How then is it possible to obtain a verification; or can we obtain absolute certainty in this matter? 26

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$$\alpha^2/p^2 + \beta^2/p_1^2 + \gamma^2/p_2^2 - 2(\beta\gamma/p_1p_2) \cos A \\ - 2(\gamma\alpha/p_2p) \cos B - 2(\alpha\beta/pp_1) \cos C = 1,$$

where α, β, γ are the distances of any point from the vertices of the triangle ABC, and p, p_1, p_2 the perpendiculars of the same triangle.

(2) Show that the curve given by the above equation is the same as that referred to by Professor Cayley in Quest. 2110, which is as follows:—Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity 105

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5042. (Editor.)—In an ellipsoid whose principal semi-axes are a, b, c , if u_1 be the sum of the n th powers of the conjugate semi-axes, and u_2 a like sum of semi-axes that are at right angles to one another, show that the minima values of u_1, u_2 are respectively

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7562. (R. Knowles, M.A.)—Prove, and verify (when $x = 0$), that

$$1 + \frac{1+x}{1.2} + \frac{(1+x)(1+2x)}{1.2.3} + \&c. \text{ ad inf. } \equiv (1+x)^{\frac{1}{2}} - 1 \quad \dots\dots\dots 97$$

7630. (W. Perryman.)—Prove, by a simple method, these differential coefficients: (1) when m is a variable and n a constant, that of $\sin(m \sin n)$ is $\sin n \cos(m \sin n)$, that of $\cos(m \sin n)$ is $-\sin n \sin(m \sin n)$, that of $\sin(m \cos n)$ is $\cos n \cos(m \cos n)$, that of $\cos(m \cos n)$ is $-\cos n \sin(m \cos n)$; (2) when m is a constant and n a variable, that of $\sin(m \sin n)$ is $m \cos n \cos(m \sin n)$, that of $\cos(m \sin n)$ is $-m \cos n \sin(m \sin n)$, that of $\sin(m \cos n)$ is $-m \sin n \cos(m \cos n)$, and that of $\cos(m \cos n)$ is $m \sin n \sin(m \cos n)$ 118

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positive integer, prove that

$$\int_0^{Kk} \frac{u}{dn^2 u} du = \frac{2}{k_1^2} KE, \quad \int_0^{2K} \frac{u}{dn^2 u} du = \frac{\pi}{2k_1^2} (1 + k_1^2) K,$$

$$\int_0^{2K} \frac{u}{(dn u)^{2m}} du = \frac{2K}{k^{2m}} \int_0^{\frac{1}{2}\pi} \Delta^{2m-1} d\phi \dots\dots\dots 119$$

11091. (Professor Neuberg.)—On donne deux droites Ox , Oy et un point. Par A , on mène une sécante quelconque qui rencontre Ox en B , Oy en C ; on partage BC au point M dans un rapport donné m/n . D'après un théorème connu, le point M décrit une hyperbole. Démontrer que l'aire du secteur décrit par le rayon vecteur OM est proportionnelle à $mn/(m+n)^2$ 57

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11857. (Professor Tissot.)—Du centre d'un cercle intérieur et bitangent à une conique, on abaisse une perpendiculaire sur une tangente à la courbe, et l'on considère les deux points de cette perpendiculaire qui se trouvent à une distance de son pied égale à la longueur de la tangente menée de ce pied au cercle. Trouver le lieu décrit par l'un de ces points, (1) lorsque la tangente est fixe et que le cercle bitangente varie, (2) lorsque, le cercle restant le même, le point de contact de la tangente se déplace sur la conique 77

11907. (Professor Morel.)—Le centre C d'un cercle se meut sur une parabole donne, et le cercle passe constamment par le foyer F de cette parabole. Trouver (1) l'enveloppe de ce cercle; (2) le lieu du point de rencontre de ce cercle (a) avec une parallèle à l'axe menée par C , (b) avec l'ordonnée de C , (c) avec la tangente à la parabole en C , (d) avec la normale menée en C 84

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 and (2) $T_{a+n} + T_{b+n} + T_{c-n-1} + T_{d+n} \equiv T_{\Sigma+n} + T_{\Sigma+n-(c+d)} + T_{\Sigma+n-(b+c)} + T_{\Sigma+n-(a+c)}$,
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13032. (M. Brierley.)—Two given circles, of centres O, P, intersect in A, B; required the length of the common chord AB when the ratio of the sum of the two lunettes O, P to the intercepted position of the two circles is as m to n 80
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13376. (Professor Neuberg.)—Soit ABCD un quadrilatère tel que $(AB)^2 + (CD)^2 = (BC)^2 + (DA)^2$. Si l'on divise chaque côté en deux segments proportionnels aux carrés des côtés adjacents, les quatre points ainsi obtenus sont concycliques..... 29

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13387. (R. F. Davis, M.A.)—Employing the notation of Salmon, prove that, if the general equation of the second degree represent a parabola, $Fx + Gy = H$ is the equation to the focal chord the tangents at whose extremities are parallel to the rectangular axes; its pole $A/2G$, $B/2F$ lies on the directrix; the focus is determined by the equations $Fx + Gy = H$ and $Gx - Fy = \frac{1}{2}(A - B)$; and the equation to the directrix is $Gx + Fy = \frac{1}{2}(A + B)$ 37

13389. (G. N. Bates.)—Prove that, if in Quest. 13332 the line AF is divided into n parts, the rest of the construction being the same,

$$AG^2 = R^2 \{n^2 + 4n + 16 - (n-4)(n^2 + 16n - 32)^{\frac{1}{2}}\} / 2(n^2 - 2n + 4) \dots 108$$

13395. (H. W. Curjel, M.A.)—Calling an infinite Euclidean hyper-space of n dimensions an S_n , show that, if pS_{n-1} 's are drawn in an S_n , they will in general divide the S_n into v_n finite and u_n infinite compartments,

where
$$v_n = \frac{(p-1)(p-2)(p-3) \dots (p-n)}{n!},$$

$$u_n = 2 \left\{ 1 + (p-1) + \frac{(p-1)(p-2)}{2!} + \&c. \text{ to } n \text{ terms} \right\} \dots 39$$

13407. (Professor Mukhopadhyay.)—Prove that (1) the number of ways in which r things may be distributed among $n+p$ persons, so that certain n of those persons may each have one at least, is

$$(n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{2!}(n+p-2)^r \dots;$$

and (2) that $S_1 = S_2 = \dots = S_{n-1} = 0$, $S_n = n!$, $S_{n+1} = (\frac{1}{2}n+p)(n+1)!$ 54

13414. (Professor Clayton.)—Prove that, if a system of conics passes through four fixed points, their director circles envelope a bicircular quartic; and, more generally, "If a system of quadric surfaces have a common curve of intersection, their director spheres touch a cyclide along a curve" 50

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13431. (Professor Crofton, F.R.S.) — (1) If t, t' are the lengths of the tangents from an external point to an ellipse, and θ be the contained angle, prove that $(1/t + 1/t') \cos \frac{1}{2}\theta = (1/r + 1/r') \cos \frac{1}{2}\alpha$,

$$(1/t - 1/t') \sin \frac{1}{2}\theta = (1/r - 1/r') \sin \frac{1}{2}\alpha,$$

where r, r' are the focal distances of the point, and α the angle they contain. Hence both the left-hand members are constant for tangents from

one point to a system of confocal ellipses. (2) If tangents be drawn from a point to two confocal ellipses, the points of contact being A, B for the outer, a, b for the inner, prove that an ellipse may be drawn with a, b as foci, and having double contact with the outer, at A and B 36

13434. (Professor Neuberg.)—En un point A d'une conique on mène la corde normale AB; en B on mène la corde BC perpendiculaire à BA, et de C on abaisse une perpendiculaire CD sur la tangente en B. Cette perpendiculaire rencontre AB en un point D. Démontrer que BD est égal au diamètre du cercle osculateur en A 31

13435. (Professor Schmitt.)—A and B are two integers, A consisting of $2m$ figures each being 1, and B consisting of m figures each being 4. Prove that $A + B + 1$ is a square 46

13436. (Professor Clayton.)—If from any point (μ) on the conic $LM = R^2$ two tangents be drawn to the conic $LM = K \cdot R^2$, the points where they meet the first conic again are of the form

$$\mu m, \mu/m, \text{ where } m + (1/m) + 2 = 4K \dots\dots\dots 53$$

13437. (Professor Bernés.)—Par le sommet de l'angle A du triangle ABC on mène deux isogonales quelconques Ax, Ay, et, par le milieu D de BC, on trace, entre ces deux droites, le segment MN partagé en D en deux parties égales. Les droites BN, CN rencontrant AM en β, γ , démontrer que les tangentes en B et C aux circonférences AB β , AC γ se rencontrent sur AN..... 77

13439. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Sum to n terms the series $s = 1 + 6 + 20 + 50 + 105 + 196 + \dots$ 56

13441. (Professor Bogle.)—A triangular piece of ground is 100 feet on each side. At each corner is a post, 40, 60, and 80 feet high, respectively. What length of rope fastened at some point within the triangle will just reach the top of each post? 45

13442. (Professor Morrell.)—Transform

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 \text{ into a product } \dots\dots\dots 52$$

13444. (Professor K. J. Sanjána, M.A.)—Examine the accuracy of the following examples in a recent treatise on trigonometry :—(1) If

$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$$

show that $\cos n\alpha + \cos n\beta + \cos n\gamma$ is zero unless n is a multiple of 3.

(2) From the expression for $x^n + 1$ in factors, deduce that

$$\begin{aligned} & \tan^{-1} \left(\frac{\sin n\theta}{1 + \cos n\theta} \right) \\ &= \tan^{-1} \frac{\sin 2\theta [1 - 2 \cos(\pi/n)]}{1 + \cos 2\theta [1 - 2 \cos(\pi/n)]} + \tan^{-1} \frac{\sin 2\theta [1 - 2 \cos(2\pi/n)]}{1 + \cos 2\theta [1 - 2 \cos(2\pi/n)]} + \dots \end{aligned}$$

[A short proof of the first example, which is correct, is required. The second example is incorrect, and when corrected leads to the evident identity $\frac{1}{2}n\theta = n \times \frac{1}{2}\theta$.] 28

13447. (D. Biddle.)—Given the perpendicular AD, and the ratios AE : EC, AF : FB formed by the feet of the other perpendiculars, construct the triangle 30

13449. (H. MacColl, B.A.)—Let p be the chance that two classes taken at random under the same arbitrary restrictions, but independently of each other, are mutually exclusive; and let the three syllogistic classes X, Y, Z , and the *universe of discourse* U , be all four thus randomly formed. Show, by elementary algebra, that the chance of *Darapti* having true premisses and a false conclusion is $3p^2 - 3p^3 + p^4$.

NOTE.—The premisses of *Darapti* are understood to assert (1) that nothing common to the classes U and Y is absent from X ; and (2) that nothing common to U and Y is absent from Z ; while the conclusion asserts that something common to U and X exists also in Z 30

13450. (W. C. Stanham.)—If A, B, C, D are concyclic points, and P, Q, R, S are the orthocentres of BCD, CDA, DAB, ABC , respectively, then P, Q, R, S are concyclic points, and A, B, C, D are the orthocentres of QRS, RSP, SPQ, PQR , respectively, and the quadrilaterals $ABCD, PQRS$ are equal and similar 45

13451. (R. Knowles, B.A.)—From a variable point T on the tangent at the vertex A of the parabola $y^2 = 4ax$, tangents TA, TP are drawn; the circle PTA cuts the parabola given in CD ; the diagonals CA, PD intersect in G . Prove that the equation to the locus of G is $ay^2 = 2x^2(x-a)$ 69

13452. (R. F. Davis, M.A.)—If P be a point within the triangle ABC such that $PA/\sin A = PB/\sin B = PC/\sin C = 2\rho$, determine ρ , and prove that $3\rho^4 = \rho^2(R^2 + \delta^2) + R^2\delta^2$ (where O = circum-centre, R = circum-radius, and $OP = \delta$). Upon what well-known line does P lie? 28

13453. (P. W. Flood.)—Find three numbers whose sum shall be a square, the sum of every two of them a square, and the first and second squares severally 53

13454. (W. J. Dobbs, M.A.)—Four coplanar forces in equilibrium act in the lines AB, BC, CD, DA , and their magnitudes are proportional to CD, DA, AB, BC , respectively. Prove that *either* the points A, B, C, D are concyclic, *or* each is the orthocentre of the triangle formed by the joins of the other three 69

13460. (I. Arnold.)—Through a given point outside or inside a given triangle to draw a right line bisecting the triangle 35

13461. (The late Professor Sylvester.—Feb. 22, 1897.)—Suppose $u_{x+1} = u_x - (u_x)^2$; prove that, if u_0 is less than unity, $u_\infty = 0$; and show more generally that, if ϕ be such an algebraical form of function that ϕx and $x - \phi x$ must each lie between 0 and 1, when x does so, then, if $u_{x+1} = \phi u_x$, and u_0 lies between 0 and 1, $u_\infty = 0$ 55

13465. (Professor H. Langhorne Orchard, M.A., B.Sc.)—If $ABCD$ is a quadrilateral inscribed in a circle, and AB, BC, CD, DA be denoted by a, b, c, d , respectively, show that $AC \cdot BD = ac + bd$, and explain the two products on the right-hand side of the equation 33

13470. (Professor W. Booth.)—In Aldis's *Algebra*, p. 353, a table is given of all the possible values of p, q, r, s, t which occur in calculating the coefficient of x^7 in $(1 + 2x + 3x^2 + 4x^3 + 5x^4)^5$; you are required to correct this table, and to show that the answer is 10040 55

13471. (Professor Shields, M.A.)—Four brothers, A., B., C., and D., bought 1350 acres of land for \$4672·50; A. and B. took $\frac{1}{2}$ of the land, paying \$2172·50 for it, of which A. paid \$693·75 and B. paid the balance. C. and D. took the remaining land, paying the balance of the money, of which C. paid \$875 and D. paid the balance, and, by reason of the difference in value of the lands, A. paid *one* dollar per acre more than B., and C. paid *one* dollar per acre more than D. How much did each pay per acre, and how much land did each get? 38

13472. (W. J. C. Miller, B.A.)—Through four given points, to draw a conic which shall intercept on a given line a chord of given length. 36

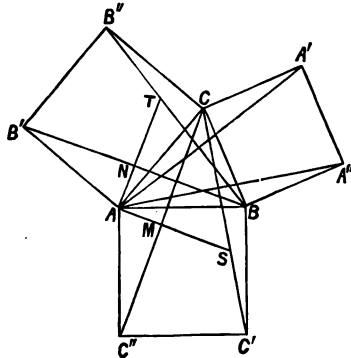
13475. (G. E. Crawford, M.A.)—Prove that the sum of the homogeneous products of the r^{th} degree of the five quantities $\alpha, \beta, \gamma, \delta, \epsilon$ is

$$\frac{\Sigma \alpha^{r+4} (\beta - \delta) (\gamma - \epsilon) (\gamma - \beta) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)}{(\alpha - \gamma) (\alpha - \epsilon) (\beta - \delta) (\beta - \alpha) (\gamma - \epsilon) (\gamma - \beta) (\delta - \alpha) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)},$$

and that the number of them is $5 \cdot 6 \cdot 7 \dots 4 + r/r!$ 66

13477. (The late Rev. W. Roberts, M.A.)—Find the parallel to $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$, which is Fresnel's "surface of elasticity"; t is also the pedal of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 - 1 = 0$ 82

13481. (W. S. Cooney, B.A. T.C.D.)—If squares $AB'B''C$, $CA'A''B$, $BC'C''A$ be described on sides of any triangle ABC , and AA' , AA'' , BB' , BB'' , CC' , CC'' be drawn from each angle of triangle to opposite corners of squares, then (1) triangle formed by intersections of AA' , BB' , CC' equals triangle formed by AA'' , BB'' , CC'' ; (2) the three circles round quadrilaterals formed by AA' , AA'' , CC' , BB'' ; BB' , BB'' , AA' , CC'' ; CC' , CC'' , BB' , AA'' have a common point O , which connects collinearly with any angle of the triangle A , the centre of opposite square, and intersection of CC'' and BB' ; (3) if perpendiculars AM , AN be drawn to CC'' and BB' , and produced to meet CC' and BB'' in S and T , then $AS = AT$, and O is centre of completed square AST , and of each square similarly described at B and C ; (4) MN is Simson's line for point A for lines BB' , CC'' , BC' , CB' 95



13487. (Rev. J. Cullen, B.A.)—Solve

$$(1 + x^2)^2 (d^2y/dx^2) + 2(1 - 3x^2)y = 0 \dots\dots\dots 38$$

13488. (Rev. T. Wiggins, B.A.)—A polygon of $2n$ sides, n of which are equal to a , and the remaining n to b , is inscribed in a circle. Show that the radius of the circle is

$$\frac{1}{2 \sin (\pi/n)} (a^2 + 2ab \cos \pi/n + b^2) \dots\dots\dots 55$$

13489. (W. J. Dobbs, M.A.)—Each of the expressions
 $x^6 - 15x^4y^2 + 15x^2y^4 - y^6, \quad 6x^5y - 20x^3y^3 + 6xy^5$
 is unaltered by the substitution of
 $\frac{x-(3)^{\frac{1}{3}}y}{2}$ for x , and $\frac{x(3)^{\frac{1}{3}}+y}{2}$ for y 47
13492. (Professor Crofton, F.R.S.)—Prove that
 $(x+y+z)^3 - 9(x^2y + y^2z + z^2x)$
 is resolvable into three linear factors 26
13493. (Professor Neuberg.)—Résoudre le système d'équations
 $\sin x \cos y = c, \quad \sin y \cos z = a, \quad \sin z \cos x = b$ 25
13495. (Professor C. Cochez.)—On donne un triangle équilatéral ABC, son centre de gravité en G. Lieu des points M tels que
 $MA \cdot MB \cdot MC = (AG)^3$ 32
13497. (Professor A. Droz-Farny.)—Si, dans un triangle ABC l'angle A est égal à 45° , la droite qui joint le centre du cercle circonscrit au pied de la hauteur abaissée de A passe par le point de Lemoine 42
13499. (Professor W. Booth.)—If $\alpha, \beta, \gamma, \delta$ be the roots of
 $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$,
 express $4(\beta\gamma - \alpha\delta) [(a + \delta)\beta\gamma - \alpha\delta(\beta + \gamma)]$
 $-(\beta + \gamma - \alpha - \delta)^2 [(a + \delta)\beta\gamma + \alpha\delta(\beta + \gamma)]$
 in terms of the coefficients 67
13507. (L. E. Reay, B.A.)—An ellipse is cut at P and P' by a tangent to its minor auxiliary circle. Prove $PP' = HP \sim SP'$ 81
13509. (W. E. Jeffares, B.A.)—Find the complete solution of
 $\{(x - \beta + \gamma)(\beta + \gamma - 2\alpha)\}^5 + \{(x - \gamma + \alpha)(\gamma + \alpha - 2\beta)\}^5$
 $+ \{(x - \alpha + \beta)(\alpha + \beta - 2\gamma)\}^5 = 0 \dots$ 108
13511. (R. Chartres.)—A circle passes through Fermat's point of an isosceles triangle and touches the base; find (1) the locus of its centre, and (2) the mean value of the circles 107
13517. (H. MacColl, B.A.)—Dividing all statements into three classes—(e) *certainties* (always true), (n) *impossibilities* (never true), (θ) *variables* (possibly true, possibly false)—to which classes must the statements α, β, u, v respectively belong, so that the implication
 $(u_\alpha + u_\beta)(\alpha_v + \beta_v) : u_v$
 (denoted by ϕ) may be a *certainty*? To which classes so that ϕ may be an *impossibility*? *Defs.*—The three symbols AB, $A + B$, A_β (or $A : B$) respectively assert that A and B are *both* true; that either A or B is true; that if A is true B is true. *Note.*—There are several solutions, of which one is that ϕ is an *impossibility* when α and u are both *certainties*, and β and v both *impossibilities* 68
13519. (Professor Crofton, F.R.S.)—Prove that—
 $f(d/dx + d/dy)e^{xy} = e^{xy} \exp. (d/ds)^2 f(s), \quad \text{where } s = x + y \dots$ 61

13520. (Professor Neuberg.)—On donne le système d'équations—
 $\sin^2 x - \sin^2 (y - z) = a$, $\sin^2 y - \sin^2 (z - x) = b$, $\sin^2 z - \sin^2 (x - y) = c$,
 calculer $\sin (x + y + z)$ 59

13521. (G. Richardson, M.A.)—The late Professor Wolstenholme, M.A., Sc.D., shortly before his death, handed to me a scrap of paper, on which he had hastily scratched the following equation in tetrahedra, saying he had proved it—

$$\frac{(a+x)^2 + u'}{\sin^2 \frac{1}{2}(A+x)} = \frac{(a-x)^2 + u'}{\sin^2 \frac{1}{2}(A-x)} = \frac{(b+y)^2 + u'}{\sin^2 \frac{1}{2}(B+y)} = \dots = u,$$

where $u = 16\Delta_0\Delta_1\Delta_2\Delta_3/9V^2$,
 $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ being the areas of the triangular faces, and V the volume of the tetrahedron, and
 $u' = (\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3)(\Delta_0 + \Delta_1 - \Delta_2 - \Delta_3)(\Delta_0 + \Delta_2 - \Delta_1 - \Delta_3)$
 $(\Delta_0 + \Delta_3 - \Delta_1 - \Delta_2)/9V^2$.

[It would appear that, in the numerators of the equation of which a proof is now required, a, b, c are the lengths of three conterminous edges, and x, y, z of the respectively opposite edges, whilst, in the denominators, A, B, C, x, y, z are the dihedral angles respectively opposite to those edges.] 102

13522. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Find the product of the series $1 + 3 + 6 + 10 + 15 + 21 + \dots$ to n terms, by the series $1 + 6 + 18 + 40 + 75 + 126 + \dots$ to n terms; and show that $n, (n+1)$, and $(n+2)$ are factors of the product 74

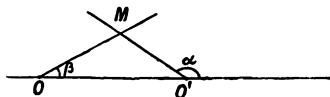
13523. (Professor Sanjána, M.A.)—Prove that

$$\frac{c_0}{m+n} - \frac{c_1}{m+3n} + \frac{c_2}{m+5n} - \dots + \frac{(-1)^l c_l}{m+(2l+1)n} \\ = \frac{2 \cdot 4 \cdot 6 \dots (2l-2) \cdot 2l \cdot n^l}{(m+n)(m+3n)(m+5n) \dots \{m+(2l+1)n\}},$$

where l, m, n are positive integers, and $c_0, c_1, c_2, \dots, c_l$ the coefficients in the expansion of $(1+x)^l$. Also deduce the identity

$$\frac{5}{9} - 4 \cdot \frac{5}{19} + 6 \cdot \frac{5}{29} - 4 \cdot \frac{5}{39} + \frac{5}{49} = \frac{5 \cdot 10 \cdot 20 \cdot 30 \cdot 40}{9 \cdot 19 \cdot 29 \cdot 39 \cdot 49} \dots\dots\dots 83.$$

13524. (Professor C. Cochez.)
 —On donne deux points O, O' . Lieu de M tel que, OM et $O'M$ tournant autour de O et de O' , on ait
 $\alpha = \frac{2}{3}\beta$.



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13526. (Professor W. Booth.)—If $\alpha, \beta, \gamma, \delta$ be the roots of the quartic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, and if all the expressions of the form $(2\alpha - \beta - \gamma)$ be taken and multiplied together, calculate the value of this product in terms of H, I, J , and G , the well-known functions of the differences of the roots 85

13528. (Professor R. W. Chapman, M.A., B.C.E.).—Two equal parabolas, A and B, are placed with their vertices together and their axes in the same straight line, but in opposite directions. If A rolls upon B, prove geometrically (a) that the focus of A traces out the directrix of B, (b) that the vertex of A traces out a cissoid, (c) that the point of intersection of the axis and directrix of A traces out the curve $r = 2a \cot \frac{1}{2}\theta$ 76

13532. (H. W. Curjel, M.A.).—Given the hypotenuse of a right-angled triangle and the length of the line bisecting the right angle, construct the triangle by means of ruler and compasses only 48

13534. (J. Woodcote West.).—Given two circles, their radii, and a point on the circumference of one, determine, geometrically, the nearest point on the circumference of the other through which a circle, passing through the given point and orthogonal to both the given circles, must pass, employing not more than five lines in your construction. Also show that the radius of the least circle orthogonal to both the given circles = $\sqrt{\frac{1}{2}S(S-R_1)(S-R_2)(S-d)}$ $\frac{2}{d}$, where d is the distance between their centres, and S half the sum of R, R_1, d 64

13535. (G. Hale Puckle, M.A.).—Find the real directrices, with the corresponding foci and eccentricity, of the conic

$$4x^2 + 24xy + 11y^2 - 4x - 2y - 9 = 0;$$

also find the axes on which (1) the real, and (2) the imaginary, foci lie. 60

13536. (Rev. W. A. Whitworth, M.A.).—If two points be taken at random on the area of a circle, their average distance apart is $\frac{128}{45\pi}$ of the radius 47

13537. (W. E. Jeffares, B.A.).—If $\tan(\psi - \phi) = \cos \alpha \tan \phi$, show that $(1 - \sin^2 \alpha \sin^2 \phi)(1 - \tan^4 \frac{1}{2}\alpha \sin^2 \psi)$ is a perfect square and independent of ψ 49

13539. (Morgan Brierley.).—Given the vertical angle, the sum of the sides including it, and the radius of the inscribed circle, to construct the triangle 107

13540. (C. Bickerdike.).—A point B moves in a right line, and A is a fixed point; the line BC passes through a fixed point D; and the angle BAC is constant. Required the locus of C 58

13541. (Charles Mitchell.).—The base of a triangle is 2 inches in length, the difference between the angles adjacent to the base is 40° . Find, geometrically, the locus of the vertex of the triangle. N.B.—A locus found by trial is not sufficient. 66

13545. (R. Chartres.).—A person can buy for 10s. an umbrella that will last him two years, or for 19s. one that will last him three years. Supposing the person to live for nearly six years, find at what per cent. the difference in the present value of the umbrellas will be (1) a maximum, (2) a minimum, and show (3) that the minimum is not the least value of the difference 49

13546. (R. F. Davis, M.A.)—Prove that the difference between the cubes of two consecutive positive integers terminates in one of the digits 1, 7, 9..... 66

13547. (R. Knowles, B.A.)—ABCD is a quadrilateral inscribed in a circle centre O; the diagonals AC and BD intersect in G; TN, T'N' are the perpendiculars from T, T', the poles of AD and BC on GO; prove that
 $TN : T'N' = TG : T'G$ 73

13552. (Professor A. Droz-Farny.)—Soient Δ et Δ' deux cercles de centres O et O' et se coupant en A et B. Une transversale variable menée par A coupe Δ en α et Δ' en α' . La parallèle menée par α à O'A coupe Δ en β , et la parallèle menée par α' à O'A coupe Δ' en β' : (1) Les trois points A, β , β' sont en ligne droite; (2) si les droites $\alpha\beta$ et $\alpha'\beta'$ se coupent en P, quel est le lieu de ce point? 106

13557. (Rev. T. Wood Robinson, R.N.)—Find the chance of throwing all sixes in r throws with n dice, and deduce the number of throws that must be made with five dice that the chance of throwing all sixes may be even 75

13562. (R. Chartres.)—If ABC be a triangle, and D a point in BC, and DA, DB, DC be denoted by x , y , z respectively, then

- (1) $by + cz > ax$; (2) $b^2y + c^2z > ax^2$;
- (3) $by^2 + cz^2$ is a minimum when A is bisected, when
- (4) $b^2y + c^2z = abc$ 65

13563. (H. W. Curjel, M.A.)—AFDB, AGECE are two intersecting straight lines; and FG, DE, BC are perpendicular to AC; and EF, DC to AB. Prove by the First Book of Euclid that DG is parallel to BE.
 64

13565. (I. Arnold.)—The ends of a cord, 10 feet long, are fastened to two points, A and B, in the same horizontal line and 6 feet apart. From the cord are suspended two weights, one of 3 lbs., the other 5 lbs. The portion of the cord between the points of suspension is also horizontal, and 3 feet perpendicular distance from AB. It is required to find the lengths of those portions of the cord between A and B and the points of suspension of the weights 79

13571. (Rev. W. Allen Whitworth, M.A.)—A stick 18 inches long is divided at random into three parts. John is to have the longest piece, Tom the next, and Sam the shortest. Determine their expectations... 99

13572. (G. Hale Puckle, M.A.)—If $\phi(xy) = 0$ and $ax + by + \gamma = 0$ represent a conic and a directrix, and (x', y') is the corresponding focus, show that $\rho^2(ax' + by' + \gamma) + \Delta^{\frac{1}{2}} = 0$, and $\rho^4\phi(x'y') = \Delta$, where
 $2\rho^2 = R - a - b$, $R^2 = (a - b)^2 + 4h^2$ 62

13573. (L. E. Reay, B.A. Suggested by Quest. 13540.)—A point B moves on a fixed conic. A and D are fixed points on the conic; BC passes through D; the angle BAC is constant. Find the locus of C... 74

13576. (Professor M. W. Crofton, F.R.S.)—If $4n - 1$ is a prime number, then $1 \cdot 2 \cdot 3 \dots (2n - 1) \pm 1$ is divisible by $4n - 1$ 89

13582. (W. J. C. Miller, B.A.)—The axes of an ellipse are 10 and 20; find the mean length of all the radius-vectors drawn from (1) the centre, (2) the focus. 80

13585. (Rev. T. Mitcheson, B.A.)—If a, b, c be the sides of a triangle, x, y, z the medians, and l, m, n the perpendiculars from the respective angular points upon the sides, then

$$4(x^2 + y^2 + z^2) = 3(l^2 + m^2 + n^2) \quad \dots\dots\dots 87$$

13592. (R. Chartres.)—Given two sides of a triangle ABC, and $\Sigma(\text{FA})$, construct the triangle, F being Fermat's point, so that $\Sigma(\text{FA})$ is a minimum 78

13602. (Rev. J. Cullen, B.A.)—Given four coplanar closed polygons whose sides are $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_r, d_1, d_2, \dots, d_s$, show that

$$(\Phi_{ab}^2 + \Psi_{ab}^2)(\Phi_{cd}^2 + \Psi_{cd}^2) = \Theta_a \Theta_b \Theta_c \Theta_d,$$

where

$$\Phi_{ab} = \Sigma a_m b_n \cos \phi_{m,n}, \quad \Psi_{ab} = \Sigma a_m b_n \sin \phi_{m,n},$$

$$\Theta_a = \Sigma a_m^2 + 2 \Sigma a_m a_n \cos \theta_{m,n},$$

excluding in each case a_p, b_q, c_r , and $d_s, \phi_{m,n}$ and $\theta_{m,n}$ being the angles between a_m and b_n, a_m and a_n 100

13606. (Professor Neuberg.)—On considère les paraboles qui touchent une droite donnée a en un point donné A et dont la tangente au sommet passe par un point donné B. Trouver l'enveloppe de l'axe 114

13616. (Jonn Sadler.)—Find the locus of the centres of all circles which touch two fixed (unequal) circles (which already do not touch one another) 96

13618. (Robert W. D. Christie.)—Where P is any odd prime whatsoever and Q an integral function of that prime, prove

$$X^2 - 5Y^2 = PQ + 1,$$

and find general integral values for X and Y 92

13619. (W. E. Jeffares, B.A.)—Prove that the circle described on the distance between the double points of the involution determined on any line by the sides of a quadrilateral inscribed in a circle cuts this circle orthogonally, and show that the circle on the third diagonal cuts orthogonally the same circle 105

13622. (R. F. Davis, M.A.)—Prove that the value of the quasi-continued fraction

$$\frac{12}{\sqrt{10} +} \frac{12}{\sqrt{10} +} \frac{12}{\sqrt{10} +} \dots \text{ad inf.}$$

is $12 \div 3.64575134 = 3.2915026$ 96

13623. (R. Chartres.)—Two random points P, Q are taken within the triangle ABC; find the mean value of the triangle APQ 92

13627. (W. J. Dobbs, M.A.)—The straight line AB is produced to P, so that rect. AP.PB = sq. on AB. In the straight line which bisects AB in C at right angles, a point Q is taken such that BQ = AP. Prove that QB bisects the angle CQP 93

13628. (J. C. Wasson.)—When a variable point moves round the circumcircle of any triangle, the motion of the Simson's line with respect to that point is identical with that of a chord of a circle one extremity of which passes over twice the arc described by the other, the arcs being measured in opposite directions..... 89

13629. (R. F. Muirhead.)—Prove that the expression

$$\alpha(\alpha-1)(\alpha-2) \dots (\alpha-r+1) + r \cdot \alpha(\alpha-1) \dots (\alpha-r+2) \beta \\ + \frac{r(r-1)}{1 \cdot 2} \alpha(\alpha-1) \dots (\alpha-r+3) \beta(\beta-1) + \dots \\ + \beta(\beta-1)(\beta-2) \dots (\beta-r+1)$$

is unaltered in value when we change α into $\alpha+1$ and β into $\beta-1$... 94

13639. (D. Biddle.)—If P be a point on the circumcircle of ABC, an equilateral triangle of side a , and if l be its perpendicular distance from the Simson-line drawn in respect of it (in the case of an equilateral triangle, any such Simson-line is equidistant from P and O the centre), prove that PA, PB, PC represent in length the three roots of the cubic equation $x^3 + qx + r = 0$, when $q = -a^2$ and $r = \pm 4R^2l = \pm \frac{4}{3}a^2l$. [All cases irreducible by Cardan's method—that is, all cases having three real roots—are included herein; for, if the value of q be supposed fixed, r can range from 0 to $\pm 2(-\frac{1}{3}q)^{\frac{1}{2}}$, whilst the three roots of the equation respectively range from 0 to $(-\frac{1}{3}q)^{\frac{1}{2}}$, from $(-q)^{\frac{1}{2}}$ to $(-\frac{1}{3}q)^{\frac{1}{2}}$, and from $(-q)^{\frac{1}{2}}$ to $2(-\frac{1}{3}q)^{\frac{1}{2}}$; two of them being of one sign, and the third of the opposite. But the finding of P, when a and l alone are given, would appear to involve the trisection of an angle] 109

13643. (R. Knowles, B.A.)—Prove that

$$1 + (n+1)x + (n+2)(x^2/1 \cdot 2) + (n+3)(x^3/1 \cdot 2 \cdot 3) + \dots = (x+n)e^x - (n-1). \\ \dots \dots \dots 113$$

13648. (R. F. Davis, M.A.)—Resolve 1,000,000,000,004 into factors

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13656. (Rev. T. Mitcheson, B.A.)—Let ABC be a triangle whose side BC is produced to D, and bisect BAC and ACD by lines meeting in E. Describe a circle about ACE, and let it cut BC in F. The bisector of FBA passes through the centre of the circle 116

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

13493. (Professor NEUBERG.)—Résoudre le système d'équations

$$\sin x \cos y = c, \quad \sin y \cos z = a, \quad \sin z \cos x = b.$$

Solution by W. E. JEFFARES, B.A.

$$\text{We have } \cos y = \frac{c}{\sin x}; \therefore \sin y = \left(1 - \frac{c^2}{\sin^2 x}\right)^{\frac{1}{2}} = \left(\frac{\sin^2 x - c^2}{\sin^2 x}\right)^{\frac{1}{2}};$$

$$\sin z = \frac{b}{\cos x}; \therefore \cos z = \left(1 - \frac{b^2}{\cos^2 x}\right)^{\frac{1}{2}} = \left(\frac{\cos^2 x - b^2}{\cos^2 x}\right)^{\frac{1}{2}};$$

$$\left(\frac{\sin^2 x - c^2}{\sin^2 x}\right)^{\frac{1}{2}} \cdot \left(\frac{\cos^2 x - b^2}{\cos^2 x}\right)^{\frac{1}{2}} = a; \quad \frac{(\sin^2 x - c^2)(\cos^2 x - b^2)}{\sin^2 x \cos^2 x} = a^2;$$

$$(\sin^2 x - c^2)(\cos^2 x - b^2) = a^2 \sin^2 x \cos^2 x;$$

$$\sin^2 x \cos^2 x - c^2 \cos^2 x - b^2 \sin^2 x + b^2 c^2 = a^2 \sin^2 x \cos^2 x;$$

$$\sin^2 2x - 2c^2(1 + \cos 2x) - 2b^2(1 - \cos 2x) + 4b^2 c^2 = a^2 \sin^2 2x;$$

$$(a^2 - 1) \cos^2 2x - 2(c^2 - b^2) \cos 2x - \{(a^2 - 1) + 2(b^2 + c^2) - 4b^2 c^2\} \cos 2x \\ = \frac{c^2 - b^2 \pm \{(a^2 + b^2 + c^2 - 1)^2 - 4a^2 b^2 c^2\}^{\frac{1}{2}}}{a^2 - 1};$$

$$\therefore x = \frac{1}{2} \cos^{-1} \frac{c^2 - b^2 \pm \{(a^2 + b^2 + c^2 - 1)^2 - 4a^2 b^2 c^2\}^{\frac{1}{2}}}{a^2 - 1}.$$

$$\text{Similarly, } y = \frac{1}{2} \cos^{-1} \frac{a^2 - c^2 \pm \{(a^2 + b^2 + c^2 - 1) - 4a^2 b^2 c^2\}^{\frac{1}{2}}}{b^2 - 1},$$

$$z = \frac{1}{2} \cos^{-1} \frac{b^2 - a^2 \pm \{(a^2 + b^2 + c^2 - 1) - 4a^2 b^2 c^2\}^{\frac{1}{2}}}{c^2 - 1}.$$

13492. (Professor CROFTON, F.R.S.)—Prove that
 $(x + y + z)^3 - 9(x^2y + y^2z + z^2x)$
 is resolvable into three linear factors.

Solution by Rev. ROBERT HARLEY, M.A., F.R.S.

Writing $x+u$ for y , and $x+v$ for z , the given function takes the form
 $(3x+u+v)^3 - 9\{x^2(x+u) + (x+u)^2(x+v) + (x+v)^2x\}$,
 which reduces to $(u+v)^3 - 9u^2v$, or $u^3 - 6u^2v + 3uv^2 + v^3$,
 a cubic in u and v . A linear factor of this cubic, found in the usual way,
 is
 $u - [2 + \{3(2+a)\}^{\frac{1}{3}} + 3/\{3(2+a)\}^{\frac{1}{3}}]v$,
 where a is an unreal cube root of unity, say $\frac{1}{2}\{-1 + \sqrt{-3}\}$; or repla-
 cing u and v by $y-x$ and $z-x$ respectively, we have

$$(y-x) - [2 + \{3(2+a)\}^{\frac{1}{3}} + 3/\{3(2+a)\}^{\frac{1}{3}}](z-x).$$

The other two linear factors of the given function are

$$(y-x) - [2 + a\{3(2+a)\}^{\frac{1}{3}} + 3a^2/\{3(2+a)\}^{\frac{1}{3}}](z-x)$$

and

$$(y-x) - [2 + a^2\{3(2+a)\}^{\frac{1}{3}} + 3a/\{3(2+a)\}^{\frac{1}{3}}](z-x).$$

4329. (W. B. DAVIS, B.A.)—The highest power of any number published is Mr. SHANKS' power of 2 to the 721st power. How may this be verified? Answer—By dividing it by 127, you will get a remainder 1, if you copy it correctly; which may be easily remembered. Objection—This only implies a "*juste rapport*" of 3 in the last place, and a do. for 2 is by no means uncommon. How then is it possible to obtain a verification; or can we obtain absolute certainty in this matter?

Solution by K. S. PUTNAM.

Such "proofs" are really only tests. If the result does not stand the test, it is wrong. If it does, it may still be wrong. Another test for any power of 2 is by "casting out the nines." Divide the index by 6, and note the remainder. If the remainder is 0, the excess in the power should be 1; if remainder is 1, excess is 2; if 2, excess is 4; if 3, excess is 8; if 4, excess is 7; if 5, excess is 5.

To prove 2^{721} , make the index even by dividing the power by 2; then extract the square root; so continue until a power is obtained either known or easily proven. This would, of course, be a tedious process.

In Vol. ii., No. 2, of the *Mathematical Visitor*, at page 32, 2^{513} is given, being 4^4 . To prove this result, Dr. MARTIN, the editor, multiplied Mr. SHANKS' 2^{508} by 2^7 , the results agreeing. This would verify Mr. SHANKS' work to that point.

Since then I have squared that result, giving

$$2^{1024} = 179,769,313,486,231,590,772,930,519,078,902,473,361,797,697; \\ 894,230,667,273,430,081,157,732,675,805,500,963,132,708,477; \\ 322,407,536,021,120,113,879,871,393,367,658,789,768,814,416, \\ 622,492,847,430,639,474,124,377,767,893,424,565,485,276,302, \\ 219,601,246,094,119,453,082,952,085,005,768,838,150,682,342, \\ 462,881,473,913,110,540,827,237,163,350,510,684,586,298,239, \\ 947,245,938,479,716,304,835,356,329,624,224,137,216.$$

I have now divided 2^{1024} by 2^{303} and obtain

$$2^{721} = 11,031,304,526,203,974,597,457,456,414,861,827,591,216,226, \\ 218,170,224,705,794,538,792,432,397,774,848,431,640,257,320, \\ 003,887,617,175,667,569,787,102,671,861,633,294,128,382,337, \\ 464,639,166,223,001,902,133,228,245,297,232,354,359,845,986, \\ 844,033,174,623,155,170,927,185,464,197,384,241,152.$$

I have not seen Mr. SHANKS' tables, but, if this agrees with his value, both his 2^{721} and my 2^{1024} must be correct.

[Mr. WOODALL, having seen the above solution, makes the following remarks upon it:—"By Algebra $2^{721}-1$ is divisible by

$$2^7-1 = 127; \text{ and } 2^{103}-1 = 10,14120,48018,25835,21197,36256,43007.$$

The former is prime, but I know no factor of the latter.

The most convenient method of obtaining 2^{721} would seem to be by raising 2^{103} to the 7th power; the best method of verification by taking the seventh root of the number which gives 2^{103} .

Mr. PUTNAM's test arises from the fact that

$$2^6 = 1 + 9 \times 7 = 1 + 9M; \text{ hence } 2^{6m} = 1 + 9M_1.$$

Another possible test arises from

$$2^{10} = 1 + 11 \times 93 = 1 + 11N; \text{ hence } 2^{10n} = 1 + 11N_1.$$

It may be noted also that $2^{730}-1$ contains the following factors—

$$3^3 \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 31 \times 37 \times 41 \times 61 \times 73 \times 97 \times 109 \times 151 \\ \times 241 \times 257 \times 331 \times 433 \times 631 \times 673 \times 1321 \times 6168 \times 23311 \\ \times 38737 \times 61681 \times 45622,84561,$$

all of which but the last are known to be primes. If this last is not a prime, it will be divisible by $120 \times l + 1$.]

4699. (Professor CROFTON, F.R.S.)—A curve rolls on a straight line; determine the nature of the motion of one of its involutes.

Solution by H. W. CURJEL, M.A.

Let P be a common point of the curve and its involute where the radius of curvature of the involute is zero. Then, if A be point of contact on the straight line when P is the point of contact of the curve, it is clear that the motion of the involute is the same as if it were made to slide through a short groove fixed at A at right angles to the given straight line. It will not limit the motion if similar grooves are fixed at all the points where the curve touches the straight line, P being the point of contact.

13452. (R. F. DAVIS, M.A.)—If P be a point within the triangle ABC such that $PA/\sin A = PB/\sin B = PC/\sin C = 2\rho$, determine ρ , and prove that $3\rho^4 = \rho^2(R^2 + \delta^2) + R^2\delta^2$ (where O = circum-centre, R = circum-radius, and OP = δ). Upon what well-known line does P lie?

Solution by G. N. BATES, B.A.

Let $AP = a\sigma$,

$\angle BPC = \alpha$, &c.

Then

$$\sum \cos^2 \alpha - 2 \cos \alpha \cos \beta \cos \gamma = 1;$$

$$\therefore \sum \left\{ \frac{(b^2 + c^2)\sigma^2 - a^2}{2bc\sigma^2} \right\} - \frac{1}{4a^2b^2c^2\sigma^6} \left[\{(b^2 + c^2)\sigma^2 - a^2\} \{(c^2 + a^2)\sigma^2 - b^2\} \{(a^2 + b^2)\sigma^2 - c^2\} \right] = 1,$$

which reduces to

$$\sigma^4 \{ \sum a^4 b^2 - \sum a^6 - 3a^2 b^2 c^2 \} + \sigma^2 \{ \sum a^4 b^2 - \sum a^6 - a^2 b^2 c^2 \} - a^2 b^2 c^2 = 0.$$

Using the identity

$$\sum a^3 \cos A \equiv abc(1 + 4 \cos A \cos B \cos C),$$

this becomes, on putting $R\sigma = \rho$,

$$\rho^4 \{ 8 \cos A \cos B \cos C - 1 \} + \rho^2 R^2 \{ 8 \cos A \cos B \cos C + 2 \} - R^4 = 0.$$

Also, since O is the centre of mean position for multiples $\sin 2A$, &c.,

$$2\rho^2(1 + 4 \cos A \cos B \cos C) = R^2 + a^2; \therefore 3\rho^4 = \rho^2(R^2 + a^2) + R^2 a^2;$$

and, by Question 13424, P lies on line joining circumcentre to orthocentre.

13444. (Professor K. J. SANJANA, M.A.)—Examine the accuracy of the following examples in a recent treatise on trigonometry:—(1) If

$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{1}{2}$$

show that $\cos n\alpha + \cos n\beta + \cos n\gamma$ is zero unless n is a multiple of 3.

(2) From the expression for $x^n + 1$ in factors, deduce that

$$\tan^{-1} \left(\frac{\sin n\theta}{1 + \cos n\theta} \right) \\ = \tan^{-1} \frac{\sin 2\theta [1 - 2 \cos(\pi/n)]}{1 + \cos 2\theta [1 - 2 \cos(\pi/n)]} + \tan^{-1} \frac{\sin 2\theta [1 - 2 \cos(2\pi/n)]}{1 + \cos 2\theta [1 - 2 \cos(2\pi/n)]} + \dots$$

[A short proof of the first example, which is correct, is required. The second example is incorrect, and when corrected leads to the evident identity $\frac{1}{2}n\theta = n \times \frac{1}{2}\theta$.]

Solution by W. J. GREENSTREET, M.A.

$$(1) \quad (\sum \cos \alpha)^2 + (\sum \sin \alpha)^2 = 3 + 2 \sum \cos(\alpha - \beta) = 0;$$

$$\therefore \sum (\cos \alpha) = \sum (\sin \alpha) = 0;$$

$$\therefore (\cos \beta + \cos \gamma)^2 + (\sin \beta + \sin \gamma)^2 = \cos^2 \alpha + \sin^2 \alpha;$$

$$\therefore 2 + 2 \cos(\beta - \gamma) = 1; \quad \therefore \beta - \gamma = \frac{3}{2}\pi.$$

Similarly,

$$\gamma - \alpha = \frac{3}{2}\pi;$$

$$\therefore \alpha = \gamma - \frac{3}{2}\pi, \quad \beta = \gamma + \frac{3}{2}\pi, \quad \gamma = \frac{1}{2}\Sigma\alpha;$$

$$\begin{aligned} \therefore \Sigma \cos n\alpha &= \cos(n\gamma - \frac{3}{2}n\pi) + \cos(n\gamma + \frac{3}{2}n\pi) + \cos n\gamma \\ &= \cos n\gamma (1 + 2 \cos \frac{3}{2}n\pi) = 0, \text{ if } n \text{ is not a multiple of } 3 \\ &= 3 \cos n\gamma, \text{ if } n \text{ is a multiple of } 3 = 3 \cos \frac{1}{2}\Sigma\alpha. \end{aligned}$$

[This question was set in Sidney Schol., 1893.]

$$(2) \text{ Let } x^n + 1 = (x - \alpha)(x - \beta)(x - \gamma) \dots$$

$$\text{Then } x^n + y^n = (x - \alpha y)(x - \beta y)(x - \gamma y) \dots,$$

$$\text{and, writing } y = \cos \theta + i \sin \theta,$$

$$x^n + \cos n\theta + i \sin n\theta = (x - \alpha \cos \theta - \alpha i \sin \theta)(x - \beta \cos \theta - \beta i \sin \theta) \dots$$

But, by a well-known theorem, if $\Pi(a_n + ib_n) = A + iB$,

$$\Sigma \tan^{-1}(b_n/a_n) = \tan^{-1}(B/A).$$

$$\therefore \Sigma \tan^{-1} \frac{\alpha \sin \theta}{\alpha \cos \theta - x} = \tan^{-1} \frac{\sin n\theta}{x^n + \cos n\theta} \text{ for all values of } x$$

But $\alpha = 1 - 2 \cos(\pi/n)$, $\beta = 1 - 2 \cos(2\pi/n)$, &c.;
therefore, writing $x = 1$,

$$\Sigma \tan^{-1} \frac{\sin \theta \{1 - 2 \cos(\pi/n)\}}{\cos \theta \{1 - 2 \cos(\pi/n)\} - x} = \tan^{-1} \frac{\sin n\theta}{1 + \cos n\theta}.$$

13376. (Professor NEUBERG.)—Soit ABCD un quadrilatère tel que $(AB)^2 + (CD)^2 = (BC)^2 + (DA)^2$. Si l'on divise chaque côté en deux segments proportionnels aux carrés des côtés adjacents, les quatre points ainsi obtenus sont concycliques.

Solution by H. W. CURJEL, M.A., and W. E. JEFFARES, B.A.

Let AB, BC, CD, DA = a, b, c, d ; and let P, Q, R, S be the points in which these sides are divided. Then $b^2 - a^2 = c^2 - d^2$; therefore the diagonals AC, BD cut at right angles.

$$\text{Also } AP = \frac{ad^2}{b^2 + d^2}, \quad AS = \frac{a^2d}{a^2 + c^2} = \frac{a^2d}{b^2 + d^2};$$

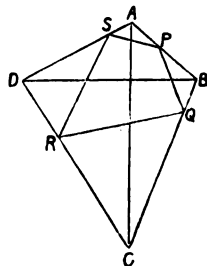
$$\text{therefore } AP : AS = d : a;$$

$$\text{therefore } \angle APS = \angle ADB;$$

$$\text{similarly } \angle BPQ = \angle BCA;$$

$$\text{and } \angle CRQ = \angle DBC \text{ and } \angle DRS = \angle DAC;$$

therefore angles APS, BPQ, CRQ, DRS are equal to two right angles; therefore angles SPQ, SRQ are equal to two right angles; therefore P, Q, R, S are concyclic.



13447. (D. BIDDLE.)—Given the perpendicular AD, and the ratios AE : EC, AF : FB formed by the feet of the other perpendiculars, construct the triangle.

Solution by (1) W. S. COONEY, B.A.; (2) L. E. REAY, B.A.; and others.

$$(1) \quad \frac{BD}{DC} = \frac{BF}{FA} \times \frac{AE}{EC} = \text{given.}$$

Draw FM and EN parallels to BC.

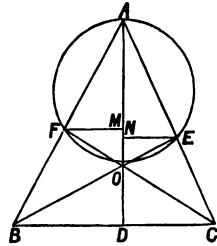
$\therefore$ M and N are given points.

$$\frac{BD}{NE} = \frac{BD}{DC} \times \frac{DC}{NE} = \text{given.}$$

$\therefore$ O is a given point.

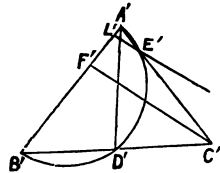
Describe a circle on AO as diameter, and from M and N draw parallels to BC.

$\therefore$ F and E are found. $\therefore$ &c.



(2) Draw any line A'B'. Divide it at F' in ratio AF : FB. Divide A'F' at L' in ratio AE : EC. Draw perpendiculars to A'B' through F' and L'.

On A'B' describe a semicircle cutting perpendicular through L' in E'. Join AE' and produce to meet the other perpendicular in C'. Join B'C'. A'B'C' is a triangle such that the feet of the perpendiculars from B' and C' divide A'B' and A'C' in the given ratios. If A'D' is drawn perpendicular to B'C', the required $\triangle ABC$ is similar to $\triangle A'B'C'$ and its sides in proportion AD : A'D' to sides of $\triangle A'B'C'$.



13449. (H. MACCOLL, B.A.)—Let p be the chance that two classes taken at random under the same arbitrary restrictions, but independently of each other, are mutually exclusive; and let the three syllogistic classes X, Y, Z, and the universe of discourse U, be all four thus randomly formed. Show, by elementary algebra, that the chance of *Darapti* having true premisses and a false conclusion is $3p^2 - 3p^3 + p^4$.

NOTE.—The premisses of *Darapti* are understood to assert (1) that nothing common to the classes U and Y is absent from X; and (2) that nothing common to U and Y is absent from Z; while the conclusion asserts that something common to U and X exists also in Z.

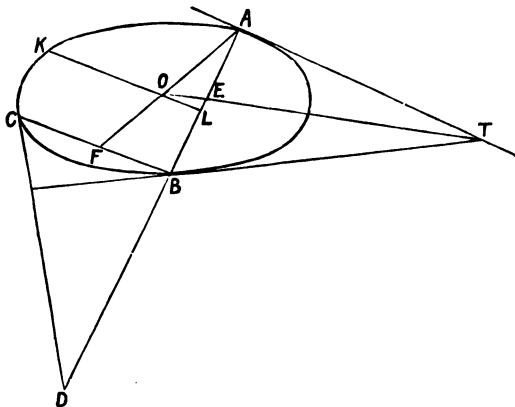
Solution by H. W. CURJEL, M.A.

The conditions require that U and Y should be mutually exclusive, and that there should be nothing common to all the three classes X, Z, U.

The chance of the former is p . The chance that there should be something common to X, Z, U is $(1-p)^3$. Therefore chance that there should be nothing common $= 3p - 3p^2 + p^3$; therefore required chance
 $= 3p^2 - 3p^3 + p^4$.

13434. (Professor NEUBERG.)—En un point A d'une conique on mène la corde normale AB ; en B on mène la corde BC perpendiculaire à BA , et de C on abaisse une perpendiculaire CD sur la tangente en B . Cette perpendiculaire rencontre AB en un point D . Démontrer que BD est égal au diamètre du cercle osculateur en A .

Solution by R. F. DAVIS, M.A.; Professor A. DROZ-FARNY; and others.



The sides of the triangle BCD being respectively perpendicular to those of the triangle ABT ,

$$BD : 2BF = AT : 2AE = OL : 2EL \dots\dots\dots (1).$$

Also

$$BF^2 : OK^2 = OA^2 - OF^2 : OA^2 = AL^2 - BL^2 : AL^2 = 4EL \cdot AE : AL^2 \dots\dots (2).$$

From (1) and (2), $BD \cdot BF : 2OK^2 = 2OL \cdot AE : AL^2$;

therefore $BD \cdot AL : 2OK^2 = 2OL \cdot AE : AL \cdot BF = 1 : 1$,

and $BD = 2OK^2/AL = \text{diameter of curvature at } A$.

3326. (By J. B. SANDERS.)—Find the velocity and periodic time of a body revolving in a circle at a distance of n radii from the earth's centre.

Solution by H. J. WOODALL, A.R.C.S.

It is proved in books on Dynamics that (for the law of the inverse square of the distance) $T^2 = Ar^3$, where T = periodic time, r the mean distance, A some constant depending on the units employed.

In the case of the moon, $T = 27.32$ days, $r = 60a$;

$$\therefore 27.32^2 = A (60a)^3.$$

In the given case $T^2 = A (na)^3$; eliminating A and a^3 and reducing, we get $T = .0588n^{\frac{1}{2}}$ nearly.

13495. (Professor C. COCHEZ.)—On donne un triangle équilatéral ABC, son centre de gravité en G. Lieu des points M tels que

$$MA \cdot MB \cdot MC = (AG)^3.$$

Solution by Rev. J. CULLEN, B.A.

Let a be the length of each of the sides, and $AM = r$. Then

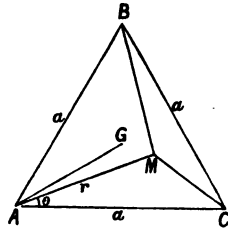
$$BM = \{a^2 + r^2 - 2ar \cos(\frac{1}{3}\pi - \theta)\}^{\frac{1}{2}},$$

$$CM = \{a^2 + r^2 - 2ar \cos \theta\}^{\frac{1}{2}},$$

$$AG = a/\sqrt{3};$$

$$\begin{aligned} \therefore r \{ (a^2 + r^2)^2 - 4ar(a^2 + r^2) \cos \frac{1}{3}\pi \cos(\frac{1}{3}\pi - \theta) \\ + 4a^2r^2 \cos(\frac{1}{3}\pi - \theta) \cos \theta \}^{\frac{1}{2}} \\ = a^3/3\sqrt{3}; \end{aligned}$$

a curve of the sixth degree.



13381. (Professor A. DROZ-FARNY.)—On considère toutes les paraboles qui passent par deux points fixes A et B et dont les directrices passent par un troisième point fixe C. Quel est le lieu de leurs foyers? Que devient le lieu si les trois points sont en ligne droite?

Solution by Rev. J. CULLEN, B.A.

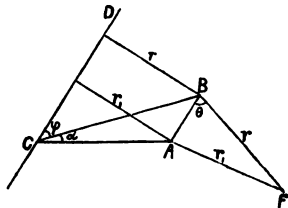
Let a , b , and c be the sides of the triangle formed by the three given points, CD a directrix, and F the focus. Then from the figure we have

$$r = a \sin \phi, \quad r_1 = b \sin(\phi + \alpha).$$

Eliminating ϕ , we get

$$b \{(a^2 - r^2)^{\frac{1}{2}} \sin \alpha + r \cos \alpha\} = ar_1.$$

Also $r_1^2 = r^2 + c^2 - 2cr \cos \theta.$



Therefore the locus of F is

$$a^2 = b^2 (r^2 + c^2 - 2cr \cos \theta) \{a^2 \sin^2 \alpha + r^2 \cos 2\alpha + r (a^2 - r^2)^{\frac{1}{2}} \sin 2\alpha\}.$$

When the three points are in a straight line $\alpha = 0$ and $a = c + b$. Hence the locus becomes $\{(b + c)^2 - b^2\} r^2 + c^2 (b + c)^2 = 2 (b + c)^2 cr \cos \theta$, the equation of a circle.

13465. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—If ABCD is a quadrilateral inscribed in a circle, and AB, BC, CD, DA be denoted by a, b, c, d , respectively, show that $AC \cdot BD = ac + bd$, and explain the two products on the right-hand side of the equation.

Solution by Rev. T. WIGGINS, B.A.

Draw BE so that

$$\angle ABE = \angle CBD.$$

Then, $\therefore \angle BAE = \angle BDC$,

$\triangle s ABE, DBC$ are similar;

$$\therefore AB : AE = BD : DC,$$

and $AB : BD = EB : BC$;

$$\therefore AB \cdot DC = DB \cdot AE.$$

Also, $\therefore \angle EBC = \angle ABD$,

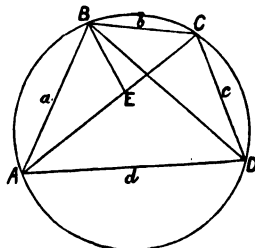
$\triangle s ABD, EBC$ are similar,

since $AB : BD = EB : BC$;

$$\therefore AD : DB = EC : BC, \text{ or } AD \cdot BC = DB \cdot EC;$$

$$\therefore AB \cdot DC + AD \cdot BC = DB \cdot AE + DB \cdot EC = DB \cdot AC;$$

$$\therefore ac + db = AC \cdot DB.$$



13259. (Professor BARISIEN.)—Résoudre un triangle, connaissant le rayon du cercle circonscrit, sa surface et celle du triangle formé par les centres des trois cercles ex-inscrits. Le problème dépend d'une équation du 3^e degré. Trouver la condition de réalité des racines.

Solution by W. E. JEFFARES, B.A.

Let a, b, c be the sides of the triangle; the equation for determining them is $x^3 - (a + b + c)x^2 + (bc + ca + ab)x - abc = 0$, a cubic equation.

Now we have only to determine $a + b + c$, $bc + ca + ab$, and abc in terms of the given quantities, viz., surface of triangle (Δ), radius of circumcircle

(R), and surface (Δ_1) of the triangle formed by joining the centres of escribed circles.

The triangle required is the orthique triangle of the triangle formed by joining the centres of its escribed circles; therefore

$$\Delta/\Delta_1 = 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$$

[see CASEY's *Trigonometry*, page 145 (353) first edition], since the angles of the triangle formed by joining the centres of escribed circles are $\frac{1}{2}(B+C)$, $\frac{1}{2}(C+A)$, $\frac{1}{2}(A+B)$, A, B, C being the angles of its orthique triangle, $\frac{\Delta}{\Delta_1} = 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = \frac{2(s-a)(s-b)(s-c)}{abc} = \frac{2\Delta^2}{abc}$.

Now $abc = 4\Delta R$;

$$\text{therefore } \frac{\Delta}{\Delta_1} = \frac{2\Delta^2}{4\Delta R s} = \frac{\Delta}{2Rs}; \quad \therefore s = \frac{\Delta_1}{2R},$$

and $a+b+c = \Delta_1/R$,

$$\Sigma(rr') = r(r' + r'' + r''') + r'r'' + r'r''' + r''r''' = (4R+r)r + s^2 = 4Rr + r^2 + s^2$$

[see CASEY's *Sequel*, page 95, Exercise 19 (sixth edition), and *Trigonometry*, page 148 (379), first edition]; but

$$\begin{aligned} \Sigma rr' &= \frac{\Delta^2}{s(s-a)} + \frac{\Delta^2}{s(s-b)} + \frac{\Delta^2}{s(s-c)} + \frac{\Delta^2}{(s-b)(s-c)} \\ &\quad + \frac{\Delta^2}{(s-c)(s-a)} + \frac{\Delta^2}{(s-a)(s-b)}, \\ \Sigma(rr') &= (s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b) + s(s-a) \\ &\quad + s(s-b) + s(s-c) s \\ &= 3S^2 - 2(a+b+c)s^2 + (bc+ca+ab) + s^2 = bc+ca+ab; \end{aligned}$$

$$\begin{aligned} \text{therefore } bc+ca+ab &= \Sigma rr' = s^2 + r^2 + 4Rr = \frac{\Delta_1^2}{4R^2} + \frac{\Delta^2}{s^2} + \frac{4R\Delta}{s} \\ &= \frac{\Delta_1^2}{4R^2} + \frac{\Delta^2 \times 4R^2}{\Delta_1^2} + \frac{8R^2\Delta}{\Delta_1} = \frac{\Delta_1^4 + 16R^4\Delta^2 + 3R^4\Delta\Delta_1}{4R^2\Delta_1^2}. \end{aligned}$$

The equation is therefore

$$x^3 - \frac{\Delta_1}{R}x^2 + \frac{\Delta_1^4 + 16R^4\Delta^2 + 32R^4\Delta\Delta_1}{4R^2\Delta_1^2}x - 4R\Delta = 0,$$

$$4R^2\Delta_1^2x^3 - 4R\Delta_1^3x^2 + (\Delta_1^4 + 32R^4\Delta\Delta_1 + 16R^4\Delta^2)x - 16R^3\Delta_1^2\Delta = 0.$$

Now the condition that roots of a cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

are real are that H is negative where H is $a_0a_2 - a_1^2$, and $G^2 + 4H^3$ is negative where

$$G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3.$$

By putting in the value of the coefficients of the given equation

$$H = \frac{(96R^4\Delta\Delta_1 + 48R^4\Delta^2 - \Delta_1^4)4R^2\Delta_1^2}{9}$$

is negative; therefore $96R^4\Delta\Delta_1 + 48R^4\Delta^2 - \Delta_1^4$ is negative (first condition).

By putting in the coefficients of the given equation

$$G = \frac{16R^3\Delta_1^5(\Delta_1^4 - 144R^4\Delta_1\Delta + 144R^4\Delta^2)}{27},$$

$$G^2 + 4H^3 = \frac{256R^6\Delta_1^6 \{ (\Delta_1^4 - 144R^4\Delta_1\Delta + 144R^4\Delta^2)^2 + (96R^4\Delta\Delta_1 + 48R^4\Delta^2 - \Delta_1^4)^3 \}}{729}$$

is negative; hence that

$$(\Delta_1^4 - 144R^4\Delta_1\Delta + 144R^4\Delta^2)^2 + (96R^4\Delta\Delta_1 + 48R^4\Delta^2 - \Delta_1^4)^3$$

is negative is the second condition.

13460. (I. ARNOLD.)—Through a given point outside or inside a given triangle to draw a right line bisecting the triangle.

Solution by W. S. COONEY, and W. E. JEFFARES, B.A.

Let P = given point. Draw CQ, making

$$\angle BCQ = \angle ACP.$$

Make rect. CP . CQ = $\frac{1}{2}$ rect. AC . CB.

On PQ describe a segment containing

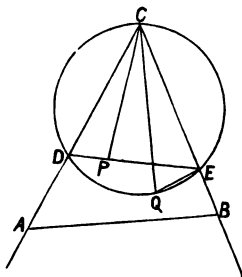
$$\angle PEQ = \angle DCQ,$$

EPD = required line. Since

$$\angle PEG = \angle DCQ,$$

CDQE is cyclic; therefore $\triangle DCP$ is similar to $\triangle QCE$.

$\therefore$ CP . CQ = CD . CE = $\frac{1}{2}$ CA . CB ;
therefore, &c.



810. (MORTIMER COLLINS.)—What is the greatest number of times in a century that February can have 5 Sundays? In what years of what century did this last happen?

Solution by K. S. PUTNAM.

February can have 5 Sundays only in leap years when February 1st comes on Sunday.

As 5 days are gained every 4 years, and the number gained must be a multiple of 7, it is evident that February will have 5 Sundays every 28th year, if leap year is not omitted. When, however, we cover a centesimal year which is not a leap year we must have $\frac{5x-1}{7}$ = a whole number, where x = number of periods of 4 years each. As $x = 3$ or 10 will answer, the number of years will be 12 or 40.

The last year in which February had 5 Sundays was 1880. This occurs

in the following years:—1604, 1632, 1660, 1688, 1728, 1756, 1784, 1824, 1852, 1880, 1920, 1948, 1976, 2004. The intervals after 2004 will be the same as those after 1604.

The greatest number of times in a century is 4. This happened in 1604, 1632, 1660, and 1688 of the 17th century, and happens on the corresponding years of every century succeeding a centesimal leap year.

13472. (EDITOR.)—Through four given points, to draw a conic which shall intercept on a given line a chord of given length.

Solution by L. E. REAY, B.A.

The system of conics cuts the fixed line in pairs of points AA' , BB' , CC' in involution.

Then we have $(ABC\dots)$ and $(A'B'C'\dots)$ homographic. From A in the direction of A' mark off AA'' equal to the given length. Similarly, mark off BB'' and CC'' .

Then $(A''B''C''\dots) = (ABC\dots) = (A'B'C'\dots)$.

Find the common points of $(A''B''\dots)$ and $(A'B'\dots)$.

Then obviously a conic through the four given points and either of these fulfils the conditions. There are two such conics, one through each point.

Taking the points A' , B' , &c., and drawing lines B' , A''' , &c., in direction A , B , &c., would give the same two conics.

13431. (Professor CROFTON, F.R.S.)—(1) If t , t' are the lengths of the tangents from an external point to an ellipse, and θ be the contained angle, prove that $(1/t + 1/t') \cos \frac{1}{2}\theta = (1/r + 1/r') \cos \frac{1}{2}\alpha$,

$$(1/t - 1/t') \sin \frac{1}{2}\theta = (1/r - 1/r') \sin \frac{1}{2}\alpha,$$

where r , r' are the focal distances of the point, and α the angle they contain. Hence both the left-hand members are constant for tangents from one point to a system of confocal ellipses. (2) If tangents be drawn from a point to two confocal ellipses, the points of contact being A , B for the outer, a , b for the inner, prove that an ellipse may be drawn with a , b as foci, and having double contact with the outer, at A and B .

Solution by R. F. DAVIS, M.A.; and Professor SANJANA, M.A.

If perpendiculars be let fall from the foci of an ellipse upon any tangent, the portion of the tangent intercepted between the feet of the perpendiculars is divided by the point of contact into segments respectively proportional to the perpendiculars. Thus, in the given case

$$r \cos \frac{1}{2}(\theta - \alpha) - t : t - r' \cos \frac{1}{2}(\theta + \alpha) = r \sin \frac{1}{2}(\theta - \alpha) : r' \sin \frac{1}{2}(\theta + \alpha),$$

which leads to

$$(1/t) \sin \theta = (1/r) \sin \frac{1}{2}(\theta + \alpha) + (1/r') \sin \frac{1}{2}(\theta - \alpha).$$

Similarly $(1/t') \sin \theta = (1/r) \sin \frac{1}{2}(\theta - \alpha) + (1/r') \sin \frac{1}{2}(\theta + \alpha).$

Then, adding and subtracting, the required result (1) follows.

The result (2) proceeds very simply and neatly from (1), in consequence of the form of the latter. For a geometrical proof see *Conics* (MILNE and DAVIS), p. 186.

1315. (EDITOR.)—Divide unity into four parts such that, if the square of one of them be diminished by four times the product of the other three, the remainder may be a rational square.

Solution by MORGAN BRIERLEY.

$$r^2 \left\{ \frac{1}{4w^2} - \frac{1}{xyz} \right\} = \frac{1}{4}.$$

Unity can be divided into four parts in several ways. Let

$$\frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = 1;$$

then, if we assume $w^2 = (\frac{1}{4})^2 = \frac{1}{16}$, we have

$$r^2 \left\{ \frac{1}{4 \cdot 4^2} - \frac{1}{3 \cdot 4 \cdot 6} \right\} = \frac{1}{4} \quad \text{and} \quad r^2 = 3^2 \cdot 4^2 = 12^2.$$

13387. (R. F. DAVIS, M.A.)—Employing the notation of SALMON, prove that, if the general equation of the second degree represent a parabola, $Fx + Gy = H$ is the equation to the focal chord the tangents at whose extremities are parallel to the rectangular axes; its pole $A/2G$, $B/2F$ lies on the directrix; the focus is determined by the equations $Fx + Gy = H$ and $Gx - Fy = \frac{1}{2}(A - B)$; and the equation to the directrix is $Gx + Fy = \frac{1}{2}(A + B)$.

Solution by REV. J. CULLEN, B.A.; and W. E. JEFFARIES, B.A.

The condition that $x - a = 0$ should touch a conic given by the general equation is $A - 2Ga + Ca^2 = 0$ or $a = A/2G$, since in a parabola $C = 0$. Therefore $x - A/2G = 0$ and, similarly, $y - B/2F = 0$ are tangents meeting at the point $P\{A/2G, B/2F\}$, and parallel to the axes. Also the director circle, being

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0,$$

here reduces to

$$Gx + Fy = \frac{1}{2}(A + B),$$

the directrix, which equation is satisfied by the foregoing values of x and y ; therefore P lies on it.

The polar of P is easily seen to be

$$(Fx + Gy - H)(gG + fF) = 0 \text{ or } Fx + Gy = H.$$

This is a focal chord (since its pole lies on the directrix), and the tangents at its extremities are $x - A/2G = 0$ and $y - B/2F = 0$. Also the focus is determined by the stated equations because $Gx - Fy = \frac{1}{2}(A - B)$ is the perpendicular on $Fx + Gy = H$ from P.

13471. (Professor SHIELDS, M.A.)—Four brothers, A., B., C., and D., bought 1350 acres of land for \$4672.50; A. and B. took $\frac{1}{2}$ of the land, paying \$2172.50 for it, of which A. paid \$693.75 and B. paid the balance. C. and D. took the remaining land, paying the balance of the money, of which C. paid \$875 and D. paid the balance, and, by reason of the difference in value of the lands, A. paid *one* dollar per acre more than B., and C. paid *one* dollar per acre more than D. How much did each pay per acre, and how much land did each get?

Solution by E. P. BARRETT, B.A.; Rev. T. WIGGINS, B.A.; and many others.

Let x = number of acres A. had; $\therefore$ B. had $450 - x$.

Let y = number of acres C. had; $\therefore$ D. had $900 - y$.

$$\therefore \$ \frac{693.75}{x} = \text{what A. paid per acre,}$$

$$\$ \frac{1478.75}{450 - x} = \text{what B. paid per acre,}$$

$$\$ \frac{875}{y} = \text{what C. paid per acre,}$$

$$\$ \frac{1625}{900 - y} = \text{what D. paid per acre;}$$

$$\therefore \frac{693.75}{x} = \frac{1478.75}{450 - x} + 1, \text{ and } \frac{875}{y} = \frac{1625}{900 - y} + 1;$$

$$\therefore x^2 - 2622.5x + 450 \times 693.75 = 0, \quad y^2 - 3400y + 900 \times 875 = 0,$$

$$(x - 125)(x - 2497.5) = 0, \quad (y - 250)(y - 3150) = 0;$$

$\therefore$ A. had 125 acres at \$5.55 per acre, B. had 325 acres at \$4.55 per acre, C. had 250 acres at \$3.50 per acre, D. had 650 acres at \$2.50 per acre.

13487. (Rev. J. CULLEN, B.A.)—Solve

$$(1 + x^2)^2 (d^2y/dx^2) + 2(1 - 3x^2)y = 0.$$

Solution by H. W. CURJEL, M.A.

Let $y = \frac{z}{1+x^2}$. Then $(1+x^2) \frac{d^2z}{dx^2} - 4x \frac{dz}{dx} = 0$;

$$\therefore z = (15x + 10x^3 + 3x^5) K + C;$$

$$\therefore y = 3Kx^3 + 7Kx + \frac{8Kx + C}{1+x^2}.$$

13395. (H. W. CURJEL, M.A.)—Calling an infinite Euclidean hyper-space of n dimensions an S_n , show that, if pS_{n-1} 's are drawn in an S_n , they will in general divide the S_n into v_n finite and u_n infinite compartments,

where
$$v_n = \frac{(p-1)(p-2)(p-3) \dots (p-n)}{n!},$$

$$u_n = 2 \left\{ 1 + (p-1) + \frac{(p-1)(p-2)}{2!} + \&c. \text{ to } n \text{ terms} \right\}.$$

Solution by the PROPOSER.

Let $v_{n,p}$ = value of v_n when there are pS_{n-1} 's. Since a system of $(p-1)S_{n-1}$'s cuts another S_{n-1} in $(p-1)S_{n-2}$'s, thus forming $v_{n-1,p-1}$ finite compartments in the last S_{n-1} , this S_{n-1} , being added to the system, cuts off $v_{n-1,p-1}$ new finite compartments from the finite and infinite compartments into which the $(p-1)S_{n-1}$'s divide the S_n .

$$\begin{aligned} \therefore v_{n,p} &= v_{n-1,p-1} + v_{n,p-1} = v_{n-1,p-1} + v_{n-1,p-2} + v_{n,p-2} \\ &= \dots \dots \dots \dots \dots \dots \dots \dots \\ &= v_{n-1,p-1} + v_{n-1,p-2} + v_{n-1,p-3} + \dots + v_{n-1,n+2} \\ &\qquad\qquad\qquad + v_{n-1,n+1} + v_{n,n+1}. \end{aligned}$$

But $v_{n,n+1} = 1$, and $v_{1,p} = p-1$, and $v_{2,p} = \frac{(p-1)(p-2)}{2!}$.

Hence, by induction,

$$v_{n,p} = \frac{(p-1)(p-2)(p-3) \dots (p-n)}{n!}.$$

Again consider an S_{n-1} which is divided into v_{n-1} finite and u_{n-1} infinite compartments by pS_{n-2} 's; let a finite part A of the S_{n-1} contain all the v_{n-1} finite compartments, and from a point B in A draw a straight line BC, whose length is great in comparison with the greatest straight line which can be drawn in A, perpendicular to the S_{n-1} .

With centre C and radius BC, describe a hypersphere H of n dimensions. Now, if pS_{n-1} 's are drawn through C and the pS_{n-2} 's, these will clearly divide the S_n into u_n infinite compartments, and also the surface of the hypersphere into u_n compartments (finite, of course), for no infinite compartments are in general created or destroyed by making the S_{n-1} 's all pass through one point, and each compartment leaves its trace on the surface of H. Of the compartments on the surface of H, there will be

two corresponding to each finite compartment of the S_{n-1} , being diametrically opposite to one another, and one corresponding to each infinite compartment of the S_{n-1} .

$$\begin{aligned}\therefore u_n &= 2v_{n-1} + u_{n-1} = 2v_{n-1} + 2v_{n-2} + u_{n-2} \\ &= \dots \dots \dots \dots \dots \dots \\ &= 2(v_{n-1} + v_{n-2} + \dots + v_1) + u_1 \\ &= 2 \left\{ 1 + (p-1) + \frac{(p-1)(p-2)}{2!} + \text{to } n \text{ terms} \right\},\end{aligned}$$

for $u_1 = 2$.

In the above, no cases have been considered in which two S_{n-1} 's are parallel, or in which $(n+1)$ S_{n-1} 's pass through a point.

628. (S. WATSON.)—From a fixed point in the circumference of a given circle, draw *any* chord, and upon it, as a diameter, let a circle be described; it is required to find the area of the curve which is the locus of the ultimate intersections of all the circles so drawn.

Solution by Professor ZERR, M.A.

The required locus for the curve $r^m = a^m \cos m\theta$, when the fixed point is the origin, is its first pedal

$$r^{m/(m+1)} = a^{m/(m+1)} \cos m/(m+1) \theta.$$

Then, if A be the area of the curve, and A' the area of the first pedal,

$$A' = (1 + \frac{1}{2}m) A.$$

When $m = 1$ the curve is the circle $r = a \cos \theta$, and its first pedal the cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$; $\therefore A' = \frac{3}{2}A = \frac{3\pi a^2}{8}$.

12874. (P. W. FLOOD.)—There is a spot in Europe where, on a certain day this year, the Sun's declination, amplitude, and co-latitude of the place were in A.P., also the declination exceeded the latitude at 6 o'clock by $1^\circ 26'$; find the time and place.

Solution by the PROPOSER.

Put $x = \cos$ of latitude, $y = \sin$ of declination, $rd = 1$, s and $c = \sin$ and $\cos$ of $1^\circ 26'$; then $x : y :: 1 : y/x = \sin$ of amplitude, and

$$1 : y :: (1-x^2)^{\frac{1}{2}} : y(1-x^2)^{\frac{1}{2}} = \sin \text{ of altitude at } 6;$$

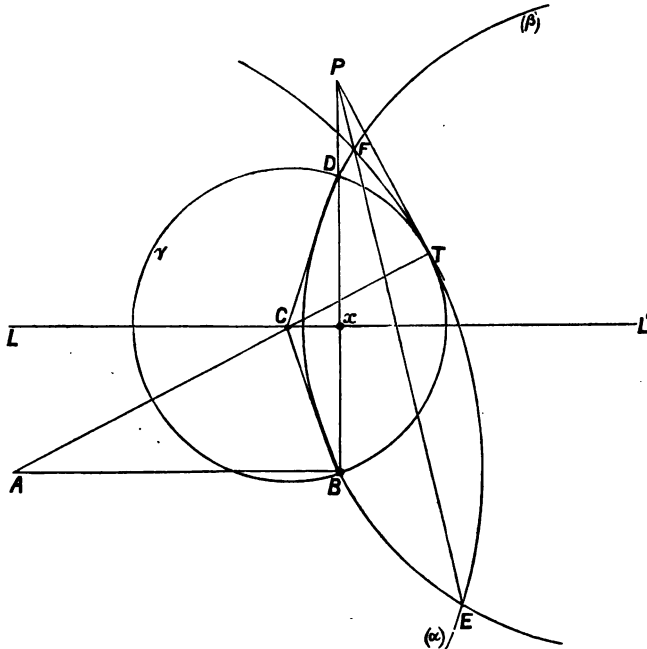
then, per question, $y(1-x^2)^{\frac{1}{2}} + x(1-y^2)^{\frac{1}{2}} = 2(1-y^2/x^2)^{\frac{1}{2}}y/x$,

and $s(1-y^2)^{\frac{1}{2}} + cy = y(1-x^2)^{\frac{1}{2}}.$

These equations solved give $x = .50879$, the $\cos$ of $59^\circ 25'$, the latitude of the place, and $y = .17737 =$ the $\sin$ of $10^\circ 13'$, answering to the 16th May, the required day.

13429. (P. W. FLOOD.)—Construct a triangle (geometrically) having given the base, the altitude, and sum of sides.

Solution by W. E. JEFFARES, B.A.



Let AB be the given base. At B erect a perpendicular Bx to AB , and equal to the given altitude. Through x draw LL' parallel to AB ; LL' is the locus of the vertex. With A as centre and the given sum of sides as radius, describe the circle (α) ; produce Bx to D until $Dx = Bx$. Now, through B and D , describe a circle touching the circle (α) as follows. Take any point on LL' as centre, and describe the circle (β) passing through B and D , and cutting circle (α) in the points E and F . Join EF , and produce it to meet BD produced in P ; from P draw PT tangent to circle (α) . Now through B, D, T describe a circle; this will be the required circle (γ) , which will touch (α) at T , for $PT^2 = PE \cdot PF = PB \times PD$ from circle (β) . Therefore PT is tangent to circle (γ) at T , and therefore the circle (γ) touches (α) . Join AT . Now the centre of (γ) lies on the line AT , and it also lies on the line LL' , since it passes through the points B and D ; therefore the intersection of AT and LL' , namely, C , is the centre

of circle (γ), and $BC = CF$. Therefore $AC + BC = AT$ equals given sum of sides; therefore ABC is the required triangle.

[Mr. BICKERDIKE solves the Question as follows:—

Let B = base, p = perpendicular, S = sum of sides, s and q = sin and cos of half the sum of the angles at the base, and x and y sin and cos half their difference. Then there results

$$\frac{s}{q} = \frac{S^2 - B^2}{2pB} = \frac{S^2 - B^2}{4a \text{ (area)}}.$$

and $qS/B = y \cos$ of half the difference. (See NEWTON's *Algebra*, p. 104.)]

13497. (Professor A. DROZ-FARNY.)—Si, dans un triangle ABC , l'angle A est égal à 45° , la droite qui joint le centre du cercle circonscrit au pied de la hauteur abaissée de A passe par le point de LEMOINE.

Solution by Professor SANJANA, M.A.

The trilinear equation of the line through circum-centre and LEMOINE's point is $\alpha \sin(B-C) + \beta \sin(C-A) + \gamma \sin(A-B) = 0$.

The coordinates of D , the foot of the perpendicular from A , are ($AD \equiv p_1$), $\alpha = 0$, $\beta = p_1 \cos C$, $\gamma = p_1 \cos B$.

Substituting these, the left side

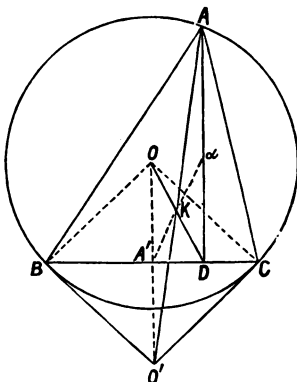
$$\begin{aligned} &= p_1 \{ \cos C \sin(C-A) + \cos B \sin(A-B) \} \\ &= \frac{1}{2} p_1 \{ \sin(2C-A) - \sin A + \sin A - \sin(2B-A) \} \\ &= p_1 \cos(B+C-A) \sin(C-B) = p_1 \cos 90^\circ \sin(C-B) = 0; \end{aligned}$$

whence the theorem.

[The PROPOSER gives the following geometrical solution of this Question:—"Soient O' le symétrique du centre O du cercle circonscrit par rapport au côté BC , A' le point milieu de BC , D le pied de la hauteur abaissée de A et α son point milieu.

Dans le triangle proposé, le quadrilatère $OBO'C$ étant un carré, O' sera le pôle de BC par rapport à la circonférence circonscrite, donc AO' est la symédiane issue de A . Mais, d'après un théorème connu, le point de LEMOINE se trouve aussi sur la droite $A'\alpha$, et par conséquent K est le point d'intersection de AO' et de $A'\alpha$.

Mais dans le trapèze $OO'DA$ les diagonales doivent se couper sur la médiane, donc O , K , et D sont en ligne droite."



13350. (Professor NATH COONDOO.)—In a triangle ABC are given the centres of the escribed circles opposite B, C, respectively, and the length of the side BC; prove that (1) A lies on a fixed straight line, (2) AB, AC are given in direction, (3) the circumscribed circle is given in magnitude, and (4) the centre of the circumscribed circle lies on a fixed circle equal to the circumscribed circle.

Solution by Rev. J. CULLEN, B.A.; W. E. JEFFARES, B.A.; and others.

Draw $O_C K$, AP , and $O_C D$ perpendicular to $O_B O_C$, BC , and AB , respectively.

(1) A clearly lies on $O_C O_B$, since common tangents meet on the join of the centres.

$$(2) \quad AO_C : O_C O_B = AH : O_B K \\ = 2S/a - S/(s-c) : S/(s-b) - S/(s-c);$$

$$\therefore a/O_C O_B = (s-b)/AO_C = AD/AO_C = \cos O_C A D, \\ \text{or} \quad \angle O_C A D = \angle O_B A C = \text{constant.}$$

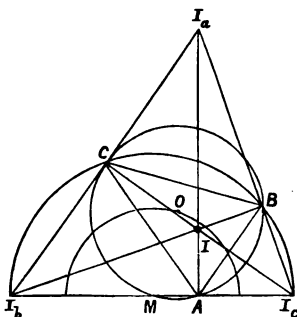
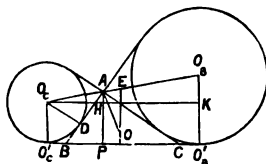
$$(3) \quad R = abc/4S = a/2 \sin A = BC/2 \sin 2O_C A D.$$

(4) Draw OE perpendicular to BC through O , the circumcentre. Then $EO_B = EO_C$.

Also $\angle EAO = \angle AEO$; $\therefore OE = OA = R$.

[Professor A. DROZ-FARNY solves the question as follows:—On sait que dans un triangle ABC les six bissectrices des angles se coupent par trois en quatre points, les centres des cercles inscrit et ex-inscrits au triangle. Dans le triangle $I_a I_b I_c$, I est l'orthocentre et A, B, C sont les pieds des hauteurs. Les sommets B et C sont sur la circonférence décrite sur $I_b I_c$ comme diamètre; comme BC est de longueur fixe les angles $BI_c C$ et $BI_b C$ sont égaux et constants. Or dans les quadrilatères inscrits $I_b B I A$ et $C I A I_b$ on a angle $BI_c C = BAI$ et $BI_b C = IAC$; on a donc: (1) A se meut sur la droite $I_b I_c$, et (2) les directions AB et AC sont fixes; (3) Angle $A = 2BAI = \text{constant}$; la circonférence circonscrite à ABC a le rayon constant déterminé par la relation $R = BC/2 \sin A$; (4) la circonférence ABC est cercle des neuf points du triangle $I_a I_b I_c$. Elle passe donc par le point milieu de $I_b I_c$. Le centre O du cercle ABC décrit une circonférence de rayon R et dont le centre est le point milieu de $I_b I_c$.

Remarque.—Les deux autres centres I_a et I décrivent deux circonférences égales de rayon égal au diamètre de ABC et passant par I_b et I_c .]



13349. (Professor FINKEL.)—Draw a line perpendicular to the base of a triangle dividing the triangle in the ratio of m to n .

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; and others.

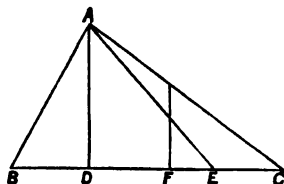
Draw AD perpendicular to the base BC of $\triangle ABC$.

Divide BC at E in the ratio $m : n$, and let E fall between D and C ; from CD cut off CF , so that

$$CF^2 = CE \cdot CD.$$

Then F is clearly the point through which the perpendicular must be drawn.

There will be two positions of F , for we may divide BC in the ratio $m : n$ or $n : m$.



Solution by Professor COLAW, A.M.

In $\triangle ABC$, let $AB = c$, $AC = b$, and $BC = a$. Represent area by k ; then

$$k = \{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}.$$

The perpendicular $CD = (2k)/c$,

$$\text{and } AD = \{b^2 - [(2k)/c]^2\}^{\frac{1}{2}}.$$

$$\text{Now } k_1 = \frac{mk}{m+n} \text{ and } k_2 = \frac{nk}{m+n};$$

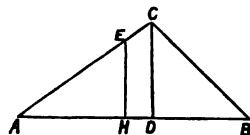
$$\therefore EH = (2mk) / \{x(m+n)\}, \text{ if we let } x = AH.$$

From similar triangles, $CD : EH = AD : AH$,

$$\text{or } (2k)/c : (2mk) / \{x(m+n)\} = 1/c (b^2c^2 - 4k^2)^{\frac{1}{2}} : x,$$

$$\text{whence } x = \{[m/(m+n)] (b^2c^2 - 4k^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

Measure distance AH and draw HE perpendicular to AB .



Solution by Professor SANJANA.

Let ABC be the triangle and AD the altitude. Take BG a mean proportional between BD and BC ; divide it at P

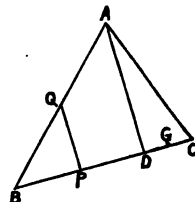
so that $BP : BG :: \sqrt{m} : \sqrt{m+n}$.

Draw parallel to AD , PQ , which shall be the line required.

$$\text{We have } \frac{BP^2}{BG^2} = \frac{m}{m+n} = \frac{BP^2}{BD \cdot BC};$$

$$\text{and } \frac{BP}{PQ} = \frac{BD}{AD};$$

$$\text{therefore } \frac{m}{m+n} \cdot \frac{BP}{PQ} = \frac{BP^2}{AD \cdot BC},$$



or
$$\frac{m}{m+n} = \frac{BP \cdot PQ}{AD \cdot BC} = \frac{2\Delta BPQ}{2\Delta ABC};$$

hence
$$\frac{m}{n} = \frac{BPQ}{ACFQ}.$$

13441. (Professor BOGLE.)—A triangular piece of ground is 100 feet on each side. At each corner is a post, 40, 60, and 80 feet high, respectively. What length of rope fastened at one point within the triangle will just reach the top of each post?

Solution by Rev. J. CULLEN, B.A.; W. E. JEFFARES, B.A.; and others.

Let P be the point, and l the length of the rope. Draw perpendiculars from P on the sides and PD (say) to meet AC in G . We have then

$$\begin{aligned} l^2 &= 60^2 + AE^2 + PE^2 \\ &= 80^2 + CE^2 + PE^2; \\ \therefore 80^2 - 60^2 &= AE^2 - (100 - AE)^2; \\ \therefore AE &= 64 \text{ and } CE = 36. \end{aligned}$$

Similarly, $CD = 26$. Also

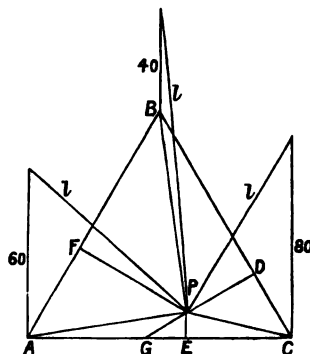
$$\begin{aligned} CD &= (CE + EG) \sin 30^\circ \\ \angle CGD = 30^\circ \text{ and } EG &= PE \cot 30^\circ; \\ \therefore PE &= 16/\sqrt{3}. \end{aligned}$$

$$\begin{aligned} \text{Hence } l^2 &= 60^2 + 64^2 + 16^2/3 \text{ or} \\ l &= 88.211 \text{ ft.} \end{aligned}$$

[Mr. CURJEL solves the Question as follows:—The perpendiculars from the required point, P , clearly divide the sides into parts 60, 40; 64, 36; 26, 74 ft. long, starting from the first post. Let x = length of perpendicular from P on the first side. Then

$$\begin{aligned} x\sqrt{3}/2 &= 44 \text{ ft.}; \therefore x = 88/\sqrt{3} \text{ ft.}; \\ \therefore \text{length of rope} &= (1600 + 3600 + 173\frac{1}{3} \text{ ft.})^{\frac{1}{2}} \\ &= \frac{1}{2} (4377 \text{ ft.})^{\frac{1}{2}} \text{ ft.} = 88.212 \text{ ft. nearly.}] \end{aligned}$$

13450. (W. C. STANHAM.)—If A, B, C, D are concyclic points, and P, Q, R, S are the orthocentres of BCD, CDA, DAB, ABC , respectively, then P, Q, R, S are concyclic points, and A, B, C, D are the orthocentres of QRS, RSP, SPQ, PQR , respectively, and the quadrilaterals $ABCD, PQRS$ are equal and similar.



Solution by H. W. CURJEL, M.A., the PROPOSER, and many others.

Since Q, P are the reflections in CD of the points in which the perpendiculars from A and B to CD meet the circle, ABPQ is a parallelogram. Similarly CDRS, BCQR, ADPS are parallelograms. Hence the theorem.

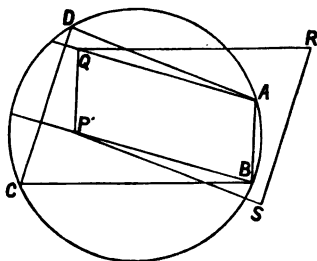
[Professor A. DROZ-FARNY solves the Question as follows :—Le théorème de Mr. STANHAM est une conséquence directe des deux théorèmes bien connus suivants :—

(i.) Si quatre points d'une hyperbole équilatère sont concycliques, l'un quelconque de ces points est diamétralement opposé de l'orthocentre du triangle des trois autres.

(ii.) L'hyperbole équilatère qui passe par les trois sommets d'un triangle passe aussi par son orthocentre.

L'hyperbole équilatère qui passe par les quatre points A, B, C, D passe aussi d'après (ii.) par P, Q, R, S. D'après (i.) les droites AP, BQ, CR, DS se coupent en parties égales au point W, centre de l'hyperbole. Les deux quadrilatères ABCD et PQRS sont en homothétie inverse ; w étant le centre et -1 le rapport l'homothétie.

L'hyperbole passant par les quatre points concycliques P, Q, R, S, et A étant diamétralement opposé de P, A est l'orthocentre du triangle QRS, &c.]



13435. (Professor SCHMITT).—A and B are two integers, A consisting of $2m$ figures each being 1, and B consisting of m figures each being 4. Prove that $A + B + 1$ is a square. _____

Solution by WILLIAM LEWIS, M.A.; T. WIGGINS, B.A.; and many others.

$$A = 1 + 10 + 10^2 + \dots + 10^{2m-1} = \frac{10^{2m} - 1}{9};$$

$$B = 4(1 + 10 + 10^2 + \dots + 10^{m-1}) = 4 \cdot \frac{10^m - 1}{9}.$$

$$\therefore A + B + 1 = \frac{10^{2m} - 1}{9} + 4 \cdot \frac{10^m - 1}{9} + 1 = \frac{10^{2m} + 4 \cdot 10^m + 4}{9}$$

$$= \left(\frac{10^m + 2}{3} \right)^2 = \text{a perfect square.}$$

[Mr. CURJEL and Mr. BATES show that the result takes the following form:—If a_m denotes m digits (each = a) following one another,

$$A + B + 1 = 1_m 5_{m-1} 6;$$

$$\therefore 9(A + B + 1) = 10_{m-1} 40_{m-1} 4 = (10_{m-1} 2)^2;$$

$$\therefore A + B + 1 = (3_{m-1} 4)^2.]$$

13489. (W. J. DOBBS, M.A.)—Each of the expressions
 $x^6 - 15x^4y^2 + 15x^2y^4 - y^6$, $6x^5y - 20x^3y^3 + 6xy^5$
 is unaltered by the substitution of

$$\frac{x - (3)^{\frac{1}{2}}y}{2} \text{ for } x, \text{ and } \frac{x(3)^{\frac{1}{2}} + y}{2} \text{ for } y.$$

Solution (1) by L. E. REAY, B.A.; (2) by Professor SANJANA, M.A.

(1) Put $\frac{x - (3)^{\frac{1}{2}}y}{2} = X$, $\frac{x(3)^{\frac{1}{2}} + y}{2} = Y$;
 $X + iY = x \left(\frac{1 + i\sqrt{3}}{2} \right) + iy \left(\frac{1 + i\sqrt{3}}{2} \right) = \omega(x + iy)$;
 $\therefore (X + iY)^6 = (x + iy)^6$;

therefore, on expanding and equating real and imaginary parts,

$$X^6 - 15X^4Y^2 + 15X^2Y^4 - Y^6 = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$$

$$\text{and } 6X^5Y - 20X^3Y^3 + 6XY^5 = 6x^5y - 20x^3y^3 + 6xy^5,$$

which shows that the expressions are unaltered by the given substitutions.

(2) Let $x = r \cos \theta$, $y = r \sin \theta$; then

$$A + iB = (x + iy)^6 = (r \cos \theta + ir \sin \theta)^6 = r^6 (\cos 6\theta + i \sin 6\theta).$$

Hence $A = r^6 \cos 6\theta$, $B = r^6 \sin 6\theta$.

Again $x_1 = \frac{1}{2}x - \frac{1}{2}\sqrt{3}y = r \left(\frac{1}{2} \cos \theta - \frac{1}{2}\sqrt{3} \sin \theta \right) = r \cos \left(\theta + \frac{1}{2}\pi \right)$;

so also $y_1 = r \sin \left(\theta + \frac{1}{2}\pi \right)$;

then $A_1 + iB_1 = r^6 \left\{ \cos 6 \left(\theta + \frac{1}{2}\pi \right) + i \sin 6 \left(\theta + \frac{1}{2}\pi \right) \right\}$.

Hence $A_1 = r^6 \{ \cos (6\theta + 2\pi) \} = A$; $B_1 = B$.

13536. (Rev. W. A. WHITWORTH, M.A.)—If two points be taken at random on the area of a circle, their average distance apart is $\frac{128}{45\pi}$ of the radius.

Solution by the PROPOSER.

(i.) Let P, Q be random points on a circle, and A a fixed point on the circumference. Let AR = r be a chord of the circle making an angle θ with the diameter AB, and let R' be a point on the circumference indefinitely near to R.

Chance of P lying on ARR' is $r^2 d\theta \div \pi a^2$, θ varying from 0 to $\frac{1}{2}\pi$. But, if P lie on ARR', the expectation of AP is $\frac{2}{3}r$. Hence

$$E(AP) = \int \frac{2}{3}r^3 d\theta \div \pi a^2 = \frac{16a}{3\pi} \int \cos^3 \theta d\theta = \frac{32a}{9\pi}.$$

(ii.) Let P be at a distance x from the centre (chance $\propto x dx$) and let Q be nearer to the centre (chance $= x^2 + a^2$).

Then by (i.) $E(PQ) = 32x + 9\pi$. But x varies from 0 to a .

Hence generally

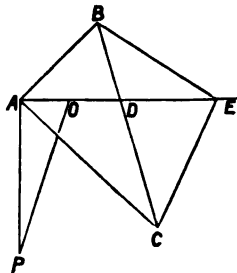
$$E(PQ) = \frac{32}{9\pi} \int_0^a x^4 dx + \int_0^a x^2 dx = \frac{128a}{45\pi}.$$

[For another Solution of this Question see Vol. LV., p. 137, under Question 11031.]

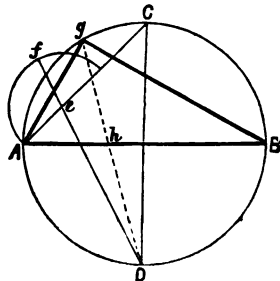
13532. (H. W. CUEJEL, M.A.)—Given the hypotenuse of a right-angled triangle and the length of the line bisecting the right angle, construct the triangle by means of ruler and compasses only.

Solution by T. M. A. COOPER, M.A.; L. E. REAY, B.A.; and others.

Let AD be the length of the given bisector. Draw AP at right angles to AD , so that $AP^2 =$ half the square on the given hypotenuse. Bisect AD at O ; produce AD to E so that $OE = OP$. Then $ED \cdot EA = AP^2$. Complete triangle ABE , so that $\angle BAE$ is half a right angle, and $BE = AP$. Join BD and produce it to meet AC , at right angles to AB at C . ABC is the required triangle. (The proof is simple. $\triangle EBD, EAB$ are similar, for $ED \cdot EA = AP^2 = BE^2$; whence $ED : EB = EB : EA$; $\therefore \angle EBD$ is half a right angle. Hence E, B, A , and C are concyclic, and, joining EC , CEB is an isosceles right-angled triangle. Therefore $BC^2 =$ twice $EB^2 =$ twice AP^2 . Hence BC must be of the given length for the hypotenuse of the required triangle. ABC must therefore be the triangle required.)



[Mr. WOODCOTE WEST solves the Question as follows:—Let AB be the given hypotenuse. On AB describe a circle, and draw the diameter CD at right angles to AB . Join AC , and make $Ae =$ one half the bisecting line. With centre e and radius Ae , describe a semicircle on AC , and draw the line Df passing through e . With centre D and radius Df , obtain the intersection g , and join Ag, gB . Then AgB is the required triangle, for, by construction, $gh = 2Ae$.]



13537. (W. E. JEFFARES, B.A.)—If $\tan(\psi - \phi) = \cos \alpha \tan \phi$, show that $(1 - \sin^2 \alpha \sin^2 \phi)(1 - \tan^4 \frac{1}{2} \alpha \sin^2 \psi)$ is a perfect square and independent of ψ .

Solution by Rev. T. ROACH, M.A.; L. E. REAY, B.A.; and others.

$$\begin{aligned}\tan(\psi - \phi) &= \cos \alpha \tan \phi; \quad \therefore \tan \psi = \frac{\tan \phi (1 + \cos \alpha)}{1 - \cos \alpha \tan^2 \phi}; \\ \therefore \sin^2 \psi &= \frac{\tan^2 \phi}{1 + \tan^2 \phi} \cdot \frac{(1 + \cos \alpha)^2}{1 + \cos^2 \alpha \tan^2 \phi} = \frac{\sin^2 \phi \cdot 4 \cos^4 \frac{1}{2} \alpha \cdot \cos^2 \phi}{1 - \sin^2 \alpha \sin^2 \phi}; \\ \therefore (1 - \sin^2 \alpha \sin^2 \phi)(1 - \tan^4 \frac{1}{2} \alpha \sin^2 \psi) &= 1 - \sin^2 \alpha \sin^2 \phi - (1 - \cos \alpha)^2 \sin^2 \phi \cos^2 \phi \\ &= 1 - \sin^2 \phi + \cos^2 \alpha \sin^2 \phi - \cos^2 \phi \sin^2 \phi \\ &\quad + 2 \cos^2 \phi \sin^2 \phi \cos \alpha - \cos^2 \alpha \sin^2 \phi \cos^2 \phi \\ &= \cos^4 \phi + 2 \cos^2 \phi \sin^2 \phi \cos \alpha + \cos^2 \alpha \sin^4 \phi = (\cos^2 \phi + \sin^2 \phi \cos \alpha)^2.\end{aligned}$$

13545. (R. CHAETRES.)—A person can buy for 10s. an umbrella that will last him two years, or for 19s. one that will last him three years. Supposing the person to live for nearly six years, find at what per cent. the difference in the present value of the umbrellas will be (1) a maximum, (2) a minimum, and show (3) that the minimum is not the least value of the difference.

Solution by the PROPOSER.

The difference in present values $= 19(1 + 1/R^3) - 10(1 + 1/R^2 + 1/R^4)$; therefore $R = 1\frac{1}{2}$, or 25 per cent. gives a maximum difference of $8\frac{1}{16}$ shillings; and $R = 1\frac{1}{3}$, or 60 per cent. gives a minimum difference of $8\frac{1}{16}$ shillings. Evidently $R = 1$, or no per cent., gives the least difference, 8 shillings.

Also the maximum value is not the greatest.

4989. (J. J. WALKER, M.A.)—Deduce the form of the Catenary from the consideration that its centre of gravity must be the lowest possible.

Solution by Rev. J. CULLEN, B.A.

Assuming the length constant and the mass proportional to the length, we have

$$\bar{y} = \int_{x_0}^{x_1} y ds / \int_{x_0}^{x_1} ds \quad \text{and} \quad \int_{x_0}^{x_1} ds = l.$$

Hence
$$V = l^{-1} y (1 + p^2)^{\frac{1}{2}} + a (1 + p^2)^{\frac{1}{2}};$$

and, since $V = Pp + C$, we have $y + al = IC(1 + p^2)^{\frac{1}{2}}$.

$$\therefore dx/dy = IC/\{(y + al)^2 - IC^2\}^{\frac{1}{2}}$$

$$\text{or } x = a \log [y + \beta + \{(y + \beta)^2 - a^2\}^{\frac{1}{2}}] + C_1,$$

a , β , and C_1 being constants depending on the conditions of the problem.

This gives the general form of the common Catenary, and more general Catenaries may be deduced with the Calculus of Variations by making the depth of the centre of gravity a maximum, as in the question.

13414. (Professor CLAYTON.)—Prove that, if a system of conics passes through four fixed points, their director circles envelope a bicircular quartic; and, more generally, "If a system of quadric surfaces have a common curve of intersection, their director spheres touch a cyclide along a curve."

Solution by the PROPOSER.

Let A, B, C, D be the points, and P, Q, R be the intersections of each pair of opposite sides of $ABCD$. Now the locus of the centres of the director circles (being the locus of centres of conics through the points) is, by a well-known theorem, a conic. Again, the triangle PQR is self-conjugate for all conics through A, B, C, D , and its circumcircle cuts all the director circles orthogonally (SALMON'S *Conics*, fifth edition, Art. 375, Ex. 2). Hence, the variable director circle cuts a fixed circle orthogonally, and has its centre on a given conic, and the envelope of such a circle is a bicircular quartic.

The more general case for quadrics appears from similar reasoning, for the director spheres of all such quadrics have a common fixed orthogonal sphere, namely, that passing through the four vertices of the *cones* of the system. Again, the locus of the centres of the director spheres (being the centres of the surfaces) is the curve of intersection of two quadrics; and since a cyclide is the envelope of a sphere whose centre describes a quadric, and which cuts a fixed sphere orthogonally, the director spheres in question plainly touch each of two fixed cyclides along their curve of intersection.

3227. (By J. B. SANDERS.)—Given the velocity, distance, and direction of projection, when the force is attractive and varies as the distance: to find the orbit.

Solution by H. J. WOODALL, A.R.C.S.

See TAIT and STEELE, *Dynamics of a Particle*, page 115.

The solution is (on the usual assumptions)

$$(x \sin \alpha - y \cos \alpha)^2 + \mu a^2 y^2 / v^2 = a^2 \sin^2 \alpha.$$

Note on a Method of Finding High Powers of 2.

By D. BIDDLE.

The following method was designed with a view of testing Mr. PUTNAM's result in his solution of Question 4329 (Vol. LXVII., p. 26). Its extreme simplicity renders it highly improbable that priority can be claimed for it, but, as it does not exist in any ordinary text-book, and is not known to occur elsewhere, it is now given for what it is worth. It depends upon the following known laws:—

(1) The final digit of powers of 2 is in succession 2, 4, 8, 6. Therefore, the final digit of 2^{4n} is always 6.

(2) In multiplying by 2 we never carry more than 1, and this occurs only when multiplying 5, 6, 7, 8, 9.

(3) The carried units by their number and position entirely govern the result. Their position is identical with that of the odd digits found in the triangular block of figures amassed by continuously doubling according to the ordinary process; and, if we divide such a block into sets of four rows each, we can regard the carried units as having a particular value according to their position in the column of four in the set, irrespective of what set it may be. For 2^{4n} , if the carried unit be in the first row of a set, its value in the result is 8; if in the second row, 4; if in the third row, 2; if in the fourth row, 6; except in the row belonging to the result itself, when the carried unit counts as 1 only. Thus, in the accompanying block of figures, taking the first column, the only odd digit is in the second row (of the set), and is represented by 4 in the result; taking the second column, we have two odd digits, in third and first rows respectively, giving $2 + 8$, represented by 0 in the result; taking the third column, we have five odd digits, in fourth, first, fourth, first, and last rows respectively, giving $6 + 8 + 6 + 8 + 1$, represented by the final digit of the sum, 9.

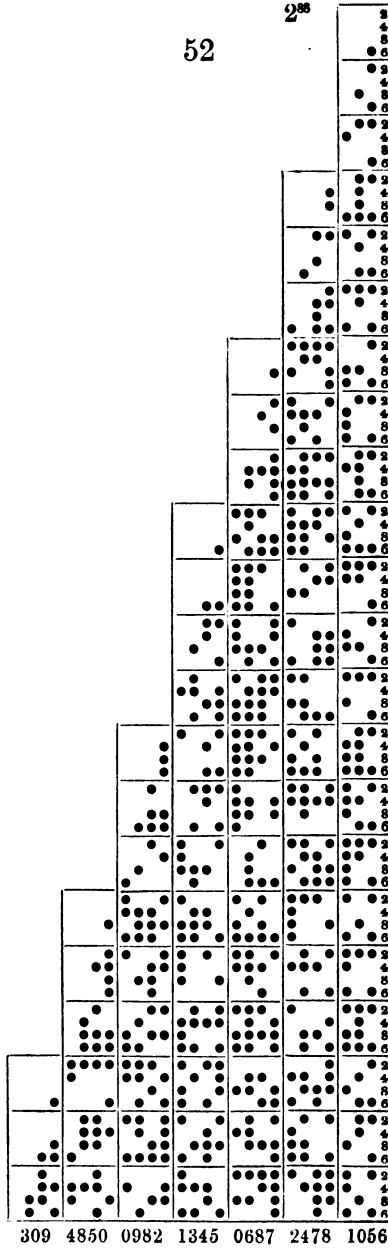
(4) It will be observed that, in every instance, the odd digit representing the carried unit has to the right of it in the row above a digit exceeding 4, and that, on the contrary, the even digits have in similar relation those of value less than 5. This law enables us to represent the position of the carried units by dots, beginning with the column next the last, which can be filled in by observation alone. Mental arithmetic does the rest:—"Twice 1 plus 1 = 3, twice 3 = 6, place dot diagonally below to the left; twice 6 gives 2, twice 2 plus 1 = 5, place dot as before; twice 5 plus 1 gives 1; twice 1 = 2, twice 2 = 4, twice 4 plus 1 = 9 in the result, thus verifying what the column of dots had previously enabled us to place; and this occurs column by column. In a very short time, by using paper ruled and cross-ruled so as to have large squares containing 4×4 smaller ones, I obtained

$$2^{88} = 309 | 4850 | 0982 | 1345 | 0687 | 2478 | 1056.$$

The verification above spoken of, however, does not protect against false position of the dots; it simply insures accuracy in summing their values. At the same time, the placing of the dots is simplicity itself, and rarely leads to error.

2
4
8
16
—
32
64
128
256
—
512
1024
2048
4096
—

2
4
8
6
—
2
4
8
6
—
2
4
8
6
—



13453. (P. W. FLOOD.)—Find three numbers whose sum shall be a square, the sum of every two of them a square, and the first and second squares severally.

Solution by the PROPOSER, and R. F. DAVIS, M.A.

Let $16x^2$, $9x^2$, and $y^2 - 10xy$ be the registered numbers. Thus four conditions of the problem are fulfilled, and it only remains to make $9x^2 - 10xy + y^2$ and $16x^2 - 10xy + y^2$ rational squares.

Let the first expression be multiplied by 16 and the second by 9.

Then $144x^2 - 160xy + 16y^2 = \square$ and $144x^2 - 90xy + 9y^2 = \square$;
their difference $= -70xy + 7y^2 = \left(24x - \frac{12y}{5}\right) \times -\frac{35y}{12}$,

a rectangle under sum and difference of their roots. Then, by a well-known rule (Duplicate Equality),

we have $\left(12x - \frac{319y}{120}\right)^2 = 144x^2 - 160xy + 16y^2$;

this solved gives $x = \frac{1385280}{1385280}$, where y may be taken 1385280 or any multiple of it. If $y = 1385280$, then $x = 128639$; hence the numbers are 148931930889, 264767877136, and 136990339200.

13442. (Professor MORRELL.)—Transform

$x^4 + y^4 + z^4 - 2y^2z^2 - 2x^2z^2 - 2x^2y^2$ into a product.

Solution by Rev. T. WIGGINS, B.A.; and many others.

Expression $= (x^2 - y^2 - z^2)^2 - 4y^2z^2 = (x^2 - y^2 - z^2 + 2yz)(x^2 - y^2 - z^2 - 2yz)$
 $= (x - y + z)(x + y - z)(x + y + z)(x - y - z).$

13436. (Professor CLAYTON.)—If from any point (μ) on the conic $LM = R^2$ two tangents be drawn to the conic $LM = K \cdot R^2$, the points where they meet the first conic again are of the form

$\mu m, \mu/m$, where $m + (1/m) + 2 = 4K$.

Solution by Rev. J. CULLEN, B.A.

Let the tangents to $LM = KR^2$ be

$m^2\mu'^2L - 2m\mu'K^{\frac{1}{2}}R + M = 0, \quad \mu'^2L - 2\mu'K^{\frac{1}{2}}R + M = 0 \dots\dots (1, 2).$

Then $(1) - (2) \equiv \mu'(1 + m)L = 2K^{\frac{1}{2}}R$;

also $(1) - m^2(2) \equiv (1 + m)M = 2m\mu'K^{\frac{1}{2}}R$;

and, since $LM = R^2$, we have $m + 1/m + 2 = 4K$.

An arbitrary chord through μ is of the form

$$\mu\mu_1 L - (\mu + \mu_1) R + M = 0.$$

Hence, comparing with (1) [or (2)], we have

$$\mu\mu_1 = m^2\mu'^2, \quad \mu + \mu_1 = 2m\mu'K^{\frac{1}{2}} = m^{\frac{1}{2}}(m+1)\mu';$$

eliminating μ' , we get $m\mu_1^2 - (1+m^2)\mu\mu_1 + m\mu^2 = 0$;

$$\therefore \mu_1 = \mu m \text{ or } \mu/m.$$

13407. (Professor MUKHOPADHYAY.)—Prove that (1) the number of ways in which r things may be distributed among $n+p$ persons, so that certain n of those persons may each have one at least, is

$$(n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{2!}(n+p-2)^r \dots;$$

and (2) that $S_1 = S_2 = \dots = S_{n-1} = 0$, $S_n = n!$, $S_{n+1} = (\frac{1}{2}n+p)(n+1)!$.

Solution by H. W. CURJEL, M.A.

The number of ways r things may be distributed among $(n+p)$ persons without restriction is $(n+p)^r$; correcting for the cases when one of the n persons has received nothing, we must subtract $n(n+p-1)^r$; here we have counted the case where two receive nothing $(2-2)$ times; therefore we must add $\frac{n(n-1)}{2!}(n+p-2)^r$, and so on.

$$\therefore S_r = (n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{2!}(n+p-2)^r - \dots,$$

for it is clear that the case where m of the n persons receive nothing is counted

$$m - \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} - \dots \text{ times} = 1 - (1-1)^m \text{ times} = \text{once}.$$

Also it is clear that, if $r < n$, $S_r = 0$;

$$\therefore S_1 = S_2 = \dots S_{n-1} = 0 \text{ and } S_n = n!.$$

If $r = n+1$, we may distribute n things to n persons in $n!$ ways and the remaining thing, which may be selected in $(n+1)$ ways, to the p persons in p ways, or to the n persons in n ways; but in the latter case each way has been counted twice;

$$\therefore S_{n+1} = (\frac{1}{2}n+p)(n+1)!.$$

[See WHITWORTH'S *Choice and Chance* (1886), Ex. 171 and Prop. XIV.]

13488. (Rev. T. WIGGINS, B.A.)—A polygon of $2n$ sides, n of which are equal to a , and the remaining n to b , is inscribed in a circle. Show that the radius of the circle is

$$\frac{1}{2 \sin(\pi/n)} (a^2 + 2ab \cos \pi/n + b^2)^{\frac{1}{2}}.$$

Solution by H. W. CURJEL, M.A.; E. P. BARRETT, B.A.; and many others.

Two adjacent sides, equal to a and b , subtend at the centre an angle $2\pi/n$, and contain an angle supplementary to π/n .

Therefore, if radius of circle = r ,

$$4r^2 \sin^2(\pi/n) = a^2 + 2ab \cos(\pi/n) + b^2.$$

13470. (Professor W. BOOTH.)—In ALDIS'S *Algebra*, p. 353, a table is given of all the possible values of p, q, r, s, t which occur in calculating the coefficient of x^7 in $(1 + 2x + 3x^2 + 4x^3 + 5x^4)^5$: you are required to correct this table, and to show that the answer is 10040.

Solution by Rev. T. WIGGINS, B.A.

$$p + q + r + s + t = 5, \quad q + 2r + 3s + 4t = 7.$$

These are satisfied by the following values:—

$p = 0;$	$q = 3;$	$r = 2;$	$s = 0;$	$t = 0;$
$p = 0;$	$q = 4;$	$r = 0;$	$s = 1;$	$t = 0;$
$p = 1;$	$q = 1;$	$r = 3;$	$s = 0;$	$t = 0;$
$p = 1;$	$q = 2;$	$r = 1;$	$s = 1;$	$t = 0;$
$p = 1;$	$q = 3;$	$r = 0;$	$s = 0;$	$t = 1;$
$p = 2;$	$q = 0;$	$r = 2;$	$s = 1;$	$t = 0;$
$p = 2;$	$q = 1;$	$r = 0;$	$s = 2;$	$t = 0;$
$p = 2;$	$q = 1;$	$r = 1;$	$s = 0;$	$t = 1;$
$p = 3;$	$q = 0;$	$r = 0;$	$s = 1;$	$t = 1.$

Therefore coefficient of x^7 is

$$\begin{aligned} & \frac{5!}{3!2!} 2^3 \cdot 3^2 + \frac{5!}{4!} 2^4 \cdot 4 + \frac{5!}{3!} 2 \times 3^3 + \frac{5!}{2!} 2^2 \cdot 3 \cdot 4 + \frac{5!}{3!} 2^3 \cdot 5 \\ & + \frac{5!}{2!2!} 3^2 \times 4 + \frac{5!}{2!} 2 \cdot 4^2 + \frac{5!}{2!} 2 \cdot 3 \cdot 5 + \frac{5!}{3!} 4 \cdot 5 \\ & = 720 + 320 + 1080 + 2880 + 800 + 1080 + 960 + 1800 + 400 \\ & = 10040. \end{aligned}$$

13461. (The late Professor SYLVESTER.—Feb. 22, 1897.)—Suppose $u_{x+1} = u_x - (u_x)^2$; prove that, if u_0 is less than unity, $u_\infty = 0$; and show more generally that, if ϕ be such an algebraical form of function that ϕx and $x - \phi x$ must each lie between 0 and 1, when x does so, then, if $u_{x+1} = \phi u_x$, and u_0 lies between 0 and 1, $u_\infty = 0$.

Solution by Professor M. W. CROFTON, F.R.S.

It is easy to see (though, perhaps, not to prove rigorously) that $u_\infty = 0$ when u_0 is between 0 and 1, as $u_1 = u_0 - u_0^2$;

$$\therefore u_0 - u_1 = u_0^2, \quad u_1 - u_2 = u_1^2, \quad u_2 - u_3 = u_2^2;$$

$$\therefore u_0 - u_r = u_0^2 + u_1^2 + u_2^2 + \dots + u_{r-1}^2;$$

so, as r increases, the excess of u_0 over u_r keeps on increasing till $u_r = 0$.

13439. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Sum to n terms the series $s = 1 + 6 + 20 + 50 + 105 + 196 + \dots$

Solution by W. E. JEFFARES, B.A.; and Rev. J. CULLEN, B.A.

See HALL and KNIGHT's *Algebra*, p. 325, Art. 396, and, much earlier, WOOD's *Algebra* (original edition).

$$S = na + \frac{n(n-1)}{2!} \Delta_1 + \frac{n(n-1)(n-2)}{3!} \Delta_2 + \frac{n(n-1)(n-2)(n-3)}{4!} \Delta_3 \\ + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} \Delta_4 + \dots;$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, &c., are the *first terms* of the orders of differences found thus—

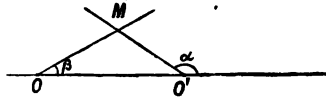
	1	6	20	50	105	196	
		5	14	30	55	91	
			9	16	25	36	
				7	9	11	
					2		
						0	

$$S = n + 5 \frac{n(n-1)}{2!} + 9 \frac{n(n-1)(n-2)}{3!} + 7 \frac{n(n-1)(n-2)(n-3)}{4!} \\ + 2 \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} \\ = \frac{n(2n^4 + 15n^3 + 40n^2 + 45n + 18)}{5!} = \frac{n(n+1)(n+2)(n+3)(2n+3)}{5!}.$$

[Mr. CURJEL solves the Question as follows:—

$$\text{The } n\text{th term} = \frac{2n(n+1)(n+2)(n+3)}{4!} - \frac{n(n+1)(n+2)}{3!}; \\ \therefore \text{sum to } n \text{ terms} = \frac{2n(n+1)(n+2)(n+3)(n+4)}{5!} - \frac{n(n+1)(n+2)(n+3)}{4!} \\ = \frac{n(n+1)(n+2)(n+3)(2n+3)}{120} \quad]$$

13524. (Professor C. COCHRAN.)
—On donne deux points O, O' . Lieu
de M tel que, OM et $O'M$ tournant
autour de O et de O' , on ait
 $\alpha = \frac{2}{3}\beta$.



Solution by Rev. J. CULLEN, B.A. ;
L. E. REAY, B.A. ; *and others.*

Let $OO' = a$.

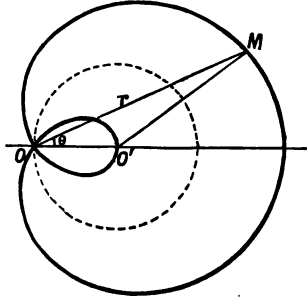
Then

$$r/a = \sin \frac{2}{3}\theta / \sin \frac{1}{3}\theta ;$$

$$\therefore r = 2a \cos \theta + a.$$

Hence the locus of M is the
outer loop of a limaçon.

The inner loop gives the locus
when $\beta = \frac{2}{3}(180^\circ - \alpha)$.



11091. (Professor NEUBERG.)—On donne deux droites Oz, Oy et un
point. Par A , on mène une sécante quelconque qui rencontre Oz en B ,
 Oy en C ; on partage BC au point M dans un rapport donné m/n . D'après
un théorème connu, le point M décrit une hyperbole. Démontrer que
l'aire du secteur décrit par le rayon vecteur OM est proportionnelle à
 $mn/(m+n)^2$.

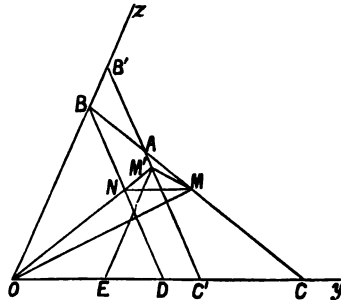
Solution by H. W. CURJEL, M.A.

Let $B'C'$ be another position
of BC , and M' the correspond-
ing position of M . Draw BD
parallel to $B'C'$, cutting OM' ,
 OC in N, D . Draw $M'E$ par-
allel to OB , cutting OC in E .
Then MN is clearly parallel to
 OC .

$$\therefore MN = m/(m+n) \cdot DC ; \text{ but}$$

$$M'E = n/(m+n) \cdot OB' ;$$

$$\therefore \triangle OMM' = mn/(m+n)^2 \\ \times \triangle B'DC.$$



Therefore each of the indefinitely small triangles with vertex O , into

which the hyperbolic sector may be divided, is $= \frac{mn}{(m+n)^2} \times$ the area of a corresponding indefinitely small triangle, whose area is independent of the ratio $m : n$.

Hence the theorem.

13540. (C. BICKERDIKE.)—A point B moves in a right line, and A is a fixed point; the line BC passes through a fixed point D; the angle BAC is constant. Required the locus of C.

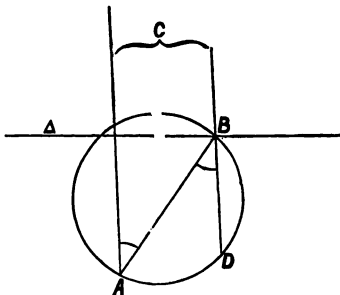
Solution by Professor A. DROZ-FARNY; Rev. J. CULLEN, B.A.; and others.

L'angle BAC étant constant, ses côtés sont les couples d'un faisceau en involution; or, les faisceaux AB et BD étant perspectifs, il en résulte que les faisceaux AC et BD sont homographiques et par conséquent le lieu de C est une conique passant par les points A et D. Les rayons AC et BD deviendront parallèles lorsque

$$\text{BAC} = \text{ABD} = \phi;$$

de là le théorème :

Faisons passer par A et D un segment de cercle capable de l'angle ϕ . La conique sera une hyperbole, une parabole ou une ellipse suivant que la circonférence coupe, touche ou ne coupe pas la droite Δ .



11326. (E. H. MOORE.)—Solve for $x_1 : x_2 : x_3$ the continued equality
 $(x_2 + x_3)(x_2 - x_3)^2 = (x_3 + x_1)(x_3 - x_1)^2 = (x_1 + x_2)(x_1 - x_2)^2$.

Solution by H. J. WOODALL, A.R.C.S.

Putting $x_2/x_1 = u$, $x_3/x_1 = v$, we have

$$(u+v)(u-v)^2 = (v+1)(v-1)^2 = (1+u)(1-u)^2;$$

therefore $(u^2 - v^2)(u - v) = (v^2 - 1)(v - 1) = (u^2 - 1)(u - 1)$;

$$\therefore \frac{u^2 - v^2}{v^2 - 1} = \frac{v - 1}{u - v}; \text{ add unity to each; } \therefore \frac{u^2 - 1}{v^2 - 1} = \frac{u - 1}{u - v} = \frac{v - 1}{u - 1};$$

$$\therefore (u - 1)^2 = (v - 1)(u - v);$$

$$\frac{u^2 - v^2}{u^2 - 1} = \frac{u-1}{u-v}; \text{ subtract unity; } \therefore \frac{1-v^2}{u^2-1} = \frac{v-1}{u-v} = \frac{1-u}{v-1};$$

$$\therefore (v-1)^2 = (u-1)(v-u); \quad \therefore (u-1)^3 + (v-1)^3 = 0.$$

Solutions are $u-1+v-1=0$; or $u+v=2$;

and $(u-1)^2 - (u-1)(v-1) + (v-1)^2 = 0$ (imaginary).

Substitute $v = 2-u$; $\therefore 2(2u-2)^2 = (3-u)(u-1)^2 = (1+u)(1-u)^2$;

$$\therefore u = 1 \text{ bis } = v \text{ or } 8 = (3-u) = (1+u), \text{ impossible.}$$

$$\therefore x_1 = x_2 = x_3.$$

13520. (Professor NEUBERG.)—On donne le système d'équations—
 $\sin^2 x - \sin^2 (y-z) = a$, $\sin^2 y - \sin^2 (z-x) = b$, $\sin^2 z - \sin^2 (x-y) = c$,
 calculer $\sin (x+y+z)$.

*Solution by the Rev. ROBERT HARLEY, M.A., F.R.S. ;
 and W. E. JEFFARES, B.A.*

Since $\sin^2 x - \sin^2 (y-z) = \sin (x+y-z) \sin (x-y+z)$,
 the given equations may be written thus:—

$$\sin (s-2z) \sin (s-2y) = a, \quad \sin (s-2x) \sin (s-2z) = b,$$

$$\sin (s-2y) \sin (s-2x) = c, \quad \text{where } s = x+y+z,$$

so that we have

$$\sin (s-2x) \sin (s-2y) \sin (s-2z) = (abc)^{\frac{1}{2}},$$

and therefore

$$\sin (s-2x) = (bc/a)^{\frac{1}{2}}, \quad \text{or } s-2x = \sin^{-1} (bc/a)^{\frac{1}{2}},$$

$$\sin (s-2y) = (ca/b)^{\frac{1}{2}}, \quad \text{or } s-2y = \sin^{-1} (ca/b)^{\frac{1}{2}},$$

$$\sin (s-2z) = (ab/c)^{\frac{1}{2}}, \quad \text{or } s-2z = \sin^{-1} (ab/c)^{\frac{1}{2}},$$

whence, by addition,

$$x+y+z = \sin^{-1} (bc/a)^{\frac{1}{2}} + \sin^{-1} (ca/b)^{\frac{1}{2}} + \sin^{-1} (ab/c)^{\frac{1}{2}},$$

$$\begin{aligned} \text{and } \sin (x+y+z), \text{ or } \sin \{ \sin^{-1} (bc/a)^{\frac{1}{2}} + \sin^{-1} (ca/b)^{\frac{1}{2}} + \sin^{-1} (ab/c)^{\frac{1}{2}} \} \\ = \{ (1/a)(b-ca)(c-ab) \}^{\frac{1}{2}} + \{ (1/b)(c-ab)(a-bc) \}^{\frac{1}{2}} + \{ (1/c)(a-bc)(b-ca) \}^{\frac{1}{2}} \\ - (abc)^{\frac{1}{2}}. \end{aligned}$$

13454. (W. J. DOBBS, M.A.)—Four coplanar forces in equilibrium act in the lines AB, BC, CD, DA, and their magnitudes are proportional to CD, DA, AB, BC, respectively. Prove that *either* the points A, B, C, D are concyclic, *or* each is the orthocentre of the triangle formed by the joins of the other three.

Solution by LIONEL E. REAY, B.A.; and Rev. J. CULLEN, B.A.

CASE I.—When there is no re-entrant angle.

Resultant of forces along AB and AD is in direction AC.

$$\therefore CD/\sin CAD = BC/\sin BAC;$$

but $CD/\sin CAD = AC/\sin ADC,$

and $BC/\sin BAC = AC/\sin ABC;$

$$\therefore \sin ABC = \sin ADC.$$

$$\therefore ABC = ADC, \text{ or } = \pi - ADC \dots (1, 2).$$

Similarly, $BCD = BAD, \text{ or } = \pi - BAD \dots \dots \dots (3, 4).$

(a) 1 with 3.—This is a possible condition of equilibrium for a *parallelogram*.

(b) 1 with 4.—This gives B and D right angles, and figure is *concyelic*.

(c) 2 with 4.—This gives figure *concyelic*.

CASE II.—Re-entrant angle at A, say.

In similar way to above, we find

$$\sin ABC = \sin ADC,$$

$$\sin BAD = \sin BCD.$$

and ABC and ADC are both acute;

$$\therefore ABC = ADC, \quad BAD = BCD$$

would give A and C coincident. $BAD = \pi - BCD$ shows that the smaller $\angle BCD$ is acute, and can only be satisfied when also $ABC = ADC$, when A is *orthocentre* of BCD. (A, p, C, q are concyclic, and angles at p and q are equal; therefore they are right angles.)

CASE III.—Crossed quadrilateral.

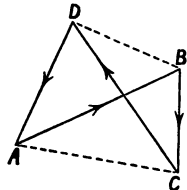
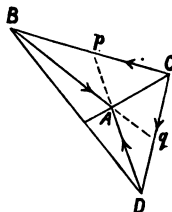
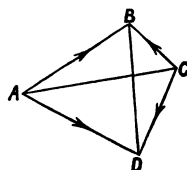
As before, we find

$$\sin ABC = \sin ADC,$$

and $\sin BAD = \sin BCD.$

(a) $ABC = \pi - ADC, \quad BAD = \pi - BCD,$
gives ADCB a *straight line*.

(b) $ABC = ADC$ gives ACBD *concyelic*.



13535. (G. HALE PUCKLE, M.A.)—Find the real directrices, with the corresponding foci and eccentricity, of the conic

$$4x^2 + 24xy + 11y^2 - 4x - 2y - 9 = 0;$$

also find the axes on which (1) the real, and (2) the imaginary, foci lie.

Solution by the PROPOSER.

By the following method the results are obtained *without reduction*.

Let the general equation $\phi(xy) = 0$ be identified, term by term, with
 $(ax + \beta y + \gamma)^2 - \rho^2 \{(x - x')^2 + (y - y')^2\} = 0$;
 then, by the resulting equations, ρ^2 is found from
 $(\rho^2 + a)(\rho^2 + b) = h^2$; and $\alpha^2 = \rho^2 + a$, $\beta^2 = \rho^2 + b$.
 Let $\phi^2 = \rho^2 + a + b$; then $\phi^2\gamma = ga + fb + \Delta^{\frac{1}{2}}$.
 The equation to the directrix may be written

$$\phi^2 ax + \phi^2 \beta y + \phi^2 \gamma = 0 \dots\dots\dots (1),$$

and $\phi^2 a = (a + \beta^2) \alpha = a\alpha + \beta h$, $\phi^2 \beta = (a^2 + b) \beta = a\alpha + \beta b$;
 hence (1) becomes $\alpha(ax + hy + g) + \beta(hx + by + f) + \Delta^{\frac{1}{2}} = 0$.

If we write $\phi(xy) = 0$, so that Δ shall be positive, and take the algebraically greater value of ρ^2 , this will represent the real directrices, the signs of α, β being determined by $a\beta = h$. The eccentricity is found from $e^2 \rho^2 = a^2 + \beta^2$. The foci are found from the equations

$$\rho^2(ax' + hy' + g) + a\Delta^{\frac{1}{2}} = 0, \quad \rho^2(hx' + by' + f) + \beta\Delta^{\frac{1}{2}} = 0;$$

and the axes of real and imaginary foci are

$$\beta(ax + hy + g) - a(hx + by + f) = 0, \quad \alpha(ax + hy + g) + \beta(hx + by + f) = 0.$$

By the above method the given equation may be written
 $(x+1)^2 + (y+1)^2 = \frac{1}{2}(3x+4y+1)^2$, or $(x-\frac{1}{3})^2 + (y-\frac{1}{3})^2 = \frac{1}{2}(3x+4y-2)^2$.

The axes of real and imaginary foci are

$$4x - 3y + 1 = 0, \quad 6x + 8y - 1 = 0.$$

See PUCKLE'S *Conic Sections*, Art. 294, Edition 1896.

[Mr. L. E. REAY solves the Question as follows:—

$$4x^2 + 24xy + 11y^2 - 4x - 2y - 9 = 0.$$

Centre of conic is $(-\frac{1}{4}, \frac{1}{2})$. Conic referred to centre is

$$4x^2 + 24xy + 11y^2 = 9;$$

$\therefore$ asymptotes are $2x + 11y = 0$, $2x + y = 0$.

Axes are $12x^2 + 7xy - 12y^2 = 0$, or $3x + 4y = 0$, $4x - 3y = 0$.

Lengths of axes are $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{2}{3}}$. Referring back to original origin, curve cuts $x = 0$ in $y = 1$ or $-\frac{1}{11}$.

Drawing the figure, we see that $3y = 4x$ is parallel to the transverse axis, the length of which is $\sqrt{\frac{2}{3}}$.

Eccentricity.— $e^2 = 1 + \frac{2}{3} = \frac{5}{3}$; $\therefore e = \sqrt{\frac{5}{3}}$.

Directrices.—If C, S, A, X, &c., have their usual meanings,

$$CX = CA/e = \frac{1}{3};$$

$\therefore$ X is $y = \frac{1}{3}(\frac{1}{3}) + \frac{1}{2} = \frac{1}{44}$, $x = \frac{1}{3}(\frac{1}{3}) - \frac{1}{2} = \frac{1}{44}$;

and, similarly, X' is $(-\frac{1}{44}, -\frac{1}{44})$; therefore directrices are

$$3(x - \frac{1}{44}) + 4(y - \frac{1}{44}) = 0 \quad \text{or} \quad 3x + 4y = \frac{1}{11},$$

$$3(x + \frac{1}{44}) + 4(y + \frac{1}{44}) = 0 \quad \text{or} \quad 3x + 4y = -\frac{1}{11}.$$

Foci.—

$$CS = e \cdot CA = \frac{1}{11};$$

$\therefore$ S is $x = \frac{2}{3}(1.5) - .1 = .8$, $y = \frac{2}{3}(1.5) + .2 = 1.4$;
and S' is $x = -\frac{2}{3}(1.5) - .1 = -1$, $y = -\frac{2}{3}(1.5) + .2 = -1$.

Ans.—From above, transverse axis $4x - 3y + 1 = 0$, and the conjugate axis is $3x + 4y = \frac{1}{2}$.

13572. (G. HALK PUCKLE, M.A.).—If $\phi(xy) = 0$ and $ax + \beta y + \gamma = 0$ represent a conic and a directrix, and (x', y') is the corresponding focus, show that $\rho^2(ax' + \beta y' + \gamma) + \Delta^{\frac{1}{2}} = 0$, and $\rho^4\phi(x'y') = \Delta$, where $2\rho^2 = R - a - b$, $R^2 = (a - b)^2 + 4h^2$.

Solutions (1) *by the PROPOSER*, (2) *by Professor SANJANA, M.A.*

(1) The first equation, and also $\rho^2(gx' + fy' + c) + \gamma\Delta^{\frac{1}{2}} = 0$, are obtained in the same manner as the equations for finding the foci in Quest. 13535 (see above); then

$$\begin{aligned}\phi(x'y') &= (ax' + hy' + g)x' + (hx' + by' + f)y' + gx' + fy' + c \\ &= -\{(\alpha\Delta^{\frac{1}{2}}x' + \beta\Delta^{\frac{1}{2}}y' + \gamma\Delta^{\frac{1}{2}})/\rho^2\} = -(\Delta^{\frac{1}{2}}/\rho^2)(ax' + \beta y' + \gamma) \\ &= \Delta/\rho^4.\end{aligned}$$

(2) Professor SANJANA solves the question as follows:—The coordinates of the focus satisfy

$$(Cx - G)^2 = \frac{1}{2}\Delta(R + a - b), \quad (Cy - F)^2 = \frac{1}{2}\Delta(R - b - a)$$

(See SALMON, sixth edition, p. 239).

Solving, we get, in the present case,

$$Cx' - G = \Delta^{\frac{1}{2}}(\rho^2 + a)^{\frac{1}{2}}, \quad Cy' - F = \Delta^{\frac{1}{2}}(\rho^2 + b)^{\frac{1}{2}}$$

Hence $aCx' + hCy' + gC = aG + hF + gC + \Delta^{\frac{1}{2}}\{a(\rho^2 + a)^{\frac{1}{2}} + h(\rho^2 + b)^{\frac{1}{2}}\}$,

$$\therefore ax' + hy' + g \equiv \frac{1}{2}\{d\phi(x'y')\}/dx' = \Delta^{\frac{1}{2}}/C\{a(\rho^2 + a)^{\frac{1}{2}} + h(\rho^2 + b)^{\frac{1}{2}}\}.$$

So also $hx' + by' + f \equiv \frac{1}{2}\{d\phi(x'y')\}/dy' = \Delta^{\frac{1}{2}}/C\{b(\rho^2 + b)^{\frac{1}{2}} + h(\rho^2 + a)^{\frac{1}{2}}\}$.

But, $x'y'$ being the focus, $\{d\phi(x'y')\}/dx' \cdot \{d\phi(x'y')\}/dy' = 4h\phi(x'y')$.

Thus $h\phi(x'y') = \Delta/C^2\{h(a\rho^2 + a^2 + b\rho^2 + b^2) + (ab + h^2)(\rho^4 + \rho^2a + \rho^2b + ab)^{\frac{1}{2}}\}$.

$$\text{Now} \quad (2\rho^2 + a + b)^2 = R^2 = (a - b)^2 + 4h^2,$$

$$\therefore \rho^4 + \rho^2a + \rho^2b = -(ab - h^2);$$

hence $h\phi(x'y') = \Delta h/C^2\{\rho^2(a + b) + a^2 + b^2 + ab + h^2\}$;

$$\therefore \phi(x'y') = \Delta/C^2\{\rho^2(a + b) + (a + b)^2 - (ab - h^2)\}$$

$$= \Delta/C^2\{(a + b)(h^2 - ab)/\rho^2 - (ab - h^2)\}$$

$$= \Delta/C\{-(a + b)/\rho^2 - 1\},$$

$$\therefore \rho^2\phi(x'y') = \Delta/C(-\rho^2 - a - b),$$

$$= \Delta/C\{+(ab - h^2)/\rho^2\}; \quad \therefore \rho^4\phi(x'y') = \Delta.$$

Hence $\rho^2\{\phi(x'y')\}^{\frac{1}{2}} + \Delta^{\frac{1}{2}} = 0$.

But, as $ax + \beta y + \gamma = 0$ is the polar of the focus $x'y'$, therefore

$$\alpha = \frac{1}{2}d\phi/dx', \quad \beta = \frac{1}{2}d\phi/dy', \quad \gamma = \frac{1}{2}d\phi/dx';$$

$$\therefore ax' + \beta y' + \gamma = \frac{1}{2}x' d\phi/dx' + \frac{1}{2}y' d\phi/dy' + gx' + fy' + c = \phi(x'y').$$

Thus $\rho^2(ax' + \beta y' + \gamma)^{\frac{1}{2}} + \Delta^{\frac{1}{2}} = 0.$

13519. (Professor CROFTON, F.R.S.)—Prove that—

$$f(d/dx + d/dy) e^{xy} = e^{xy} \exp. (d/ds)^2 f(s), \quad \text{where } s = x + y.$$

Solutions (1) by Rev. S. SIRCUM, M.A.; (2) by the PROPOSER.

(1) Let $v = xy$; then $d/dx = ds/dx \cdot d/ds + dv/dx \cdot d/dv$;
so for d/dy ; then $(d/dx + d/dy) e^{xy} = e^v (x + y) = e^v s$;
 $(d/dx + d/dy)^2 e^{xy} = e^v (s + 2d/ds) s = e^v \exp. (d/ds)^2 s^2.$

Then, if $(d/dx + d/dy)^{n-1} e^{xy} = e^{xy} \exp. (d/ds)^2 s^{n-1}$
 $(d/dx + d/dy)^n e^{xy} = e^{xy} (s + 2d/ds) \exp. (d/ds)^2 s^{n-1}$;

but $[x + f'(D)] e^{f(D)} \phi(x) = e^{f(D)} x \phi(x)$,

whence $(d/dx + d/dy)^n e^{xy} = e^{xy} \exp. (d/ds)^2 s^n$,

and generally $f(d/dx + d/dy) e^{xy} = e^{xy} \exp. (d/ds)^2 f(s).$

(2) Let $s = x + y$, $\delta = x - y$; then, if we change the independent variables to s, δ ,

$$d/dx + d/dy = 2d/ds, \quad d/dx - d/dy = 2d/d\delta;$$

$$\therefore f(d/dx + d/dy) e^{xy} = f(2d/ds) \exp. (\frac{1}{2}s^2 - \frac{1}{2}\delta^2).$$

It is known that

$$\begin{aligned} f(D + ax) &= \exp. (-\frac{1}{2}ax^2) f(D) \exp. (\frac{1}{2}ax^2) \\ &= \exp. (\frac{1}{2}a^{-1}D^2) f(ax) \exp. (-\frac{1}{2}a^{-1}D^2). \end{aligned}$$

Take 1 for operand, put $a = \frac{1}{2}$,

$$\exp. (-\frac{1}{2}s^2) f(2d/ds) \exp. (\frac{1}{2}s^2) = \exp. (d/ds)^2 f(s);$$

$$\begin{aligned} \therefore f(d/dx + d/dy) e^{xy} &= \exp. (-\frac{1}{2}\delta^2) \exp. (\frac{1}{2}s^2) \exp. (d/ds)^2 f(s) \\ &= e^{xy} \exp. (d/ds)^2 f(s). \end{aligned}$$

[Mr. CULLEN solves the Question as follows:—

Taking a term in the function of the form $(d/dx + d/dy)^n e^{xy}$, expanding, and observing that

$$(d/dx)^p (d/dy)^q e^{xy} = (d/dx)^p e^{xy} x^q = e^{xy} (d/dx + y)^p x^q,$$

we easily find $(d/dx + d/dy)^n e^{xy}$

$$= e^{xy} [(x+y)^n + n(n-1)(x+y)^{n-2} + n(n-1)(n-2)(n-3)(x+y)^{n-4}/2! + \dots]$$

$$= e^{xy} [s^n + n(n-1)s^{n-2} + n(n-1)(n-2)(n-3)s^{n-4}/2! + \dots]$$

$$= e^{xy} \exp. (d/ds)^2 s^n.$$

Collecting all such terms, we have the result in the Question.]

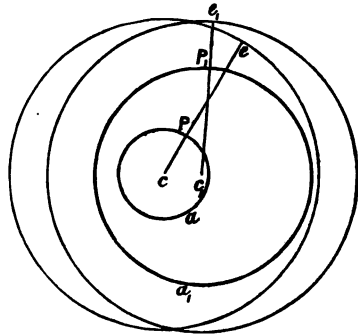
13534. (J. WOODCOTE WEST.)—Given two circles, their radii, and a point on the circumference of one, determine, geometrically, the nearest point on the circumference of the other through which a circle, passing through the given point and orthogonal to both the given circles, must pass, employing not more than five lines in your construction. Also show that the radius of the least circle orthogonal to both the given circles $= \sqrt{\{-S(S-R_1)(S-R)(S-d)\} 2/d}$, where d is the distance between their centres, and S half the sum of R, R_1, d .

Solution by the PROPOSER.

Let a, a_1 be the given circles, having centres c, c_1 , and P the given point. With radius $R+R_1$, describe concentric circles, and draw cPe . With centre e and radius d , obtain the intersection e_1 , and draw e_1c_1 intersecting a_1 in P_1 . P_1 is the required point by similar triangles. The distance of the radical axis from the nearest centre

$$= \frac{R_1^2 - R^2 - d^2}{2d};$$

hence the equation.



13563. (H. W. CURJEL, M.A.)—AFDB, AGECE are two intersecting straight lines; and FG, DE, BC are perpendicular to AC; and EF, DC to AB. Prove by the First Book of Euclid that DG is parallel to BE.

Solution by F. H. PRACHELL, B.A.; JOHN SADLER; and others.

Now, because OB is parallel to GF, therefore

$$\angle GCF = \angle GBF.$$

To each add $\angle AGF$;

$$\therefore \angle AFC = \angle AGB,$$

that is

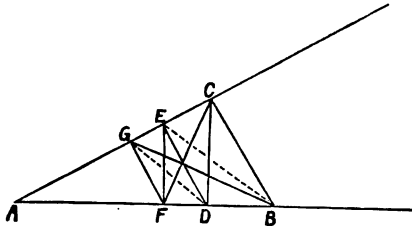
$$\angle AED = \angle AGB.$$

[For $\angle EFD = \angle EFC$.]

$$\therefore \angle AGD + \angle GED$$

$$= \angle AGD + \angle GDB;$$

therefore $\angle GED = \angle GDB$; and therefore GD is parallel to EB.



13475. (G. E. CRAWFORD, M.A.)—Prove that the sum of the homogeneous products of the r^{th} degree of the five quantities $\alpha, \beta, \gamma, \delta, \epsilon$ is

$$\frac{\Sigma \alpha^{r+4} (\beta - \delta) (\gamma - \epsilon) (\gamma - \beta) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)}{(\alpha - \gamma) (\alpha - \epsilon) (\beta - \delta) (\beta - \alpha) (\gamma - \epsilon) (\gamma - \beta) (\delta - \alpha) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)},$$

and that the number of them is $5 \cdot 6 \cdot 7 \dots 4 + r/r!$.

Solution by Rev. T. WIGGINS, B.A.; Rev. J. CULLEN, B.A.; and
H. W. CURJEL, M.A.

(1) S_r is sum of homogeneous products of r^{th} degree when

$$\frac{1}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\epsilon x)} = 1 + S_1 x + S_2 x^2 + \dots + S_r x^r + \dots$$

Assume $\frac{1}{\{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\epsilon x)\}}$
 $= A/(1-\alpha x) + B/(1-\beta x) + C/(1-\gamma x) + D/(1-\delta x) + E/(1-\epsilon x).$

Then, by partial fractions, $A = \alpha^4 / \{(a-\beta)(a-\gamma)(a-\delta)(a-\epsilon)\}$,
with similar values for B, C, D, and E.

$$\therefore \frac{1}{\{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\epsilon x)\}} = \alpha^4 / \{(a-\beta)(a-\gamma)(a-\delta)(a-\epsilon)\} (1-\alpha x)^{-1} + \dots$$

Coefficient of x^r is therefore $\Sigma \alpha^{r+4} / \{(a-\beta)(a-\gamma)(a-\delta)(a-\epsilon)\}$,
that is,

$$\frac{\Sigma \alpha^{r+4} (\beta - \delta) (\gamma - \epsilon) (\gamma - \beta) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)}{(\alpha - \gamma) (\alpha - \epsilon) (\beta - \delta) (\beta - \alpha) (\gamma - \epsilon) (\gamma - \beta) (\delta - \alpha) (\delta - \gamma) (\epsilon - \beta) (\epsilon - \delta)}.$$

(2) The number of the products is ${}_5H_r$, i.e.,

$$\frac{(5+r-1)!}{r! \cdot 4!} \quad \text{or} \quad \frac{5 \cdot 6 \cdot 7 \dots (4+r)}{r!}.$$

13546. (R. F. DAVIS, M.A.)—Prove that the difference between the cubes of two consecutive positive integers terminates in one of the digits 1, 7, 9.

Solution by Rev. J. CULLEN, B.A.; the PROPOSER; and others.

If m and $m+1$ are the integers, $(m+1)^3 - m^3 = 3m(m+1) + 1 = \text{difference}$.
Now, since the digit in which the product of any two consecutive integers between 0 and 10 terminates is 0, 2, or 6, it follows that $3m(m+1)$ terminates in 0, 6, or 8; therefore difference in 1, 7, or 9.

13541. (CHARLES MITCHELL.)—The base of a triangle is 2 inches in length, the difference between the angles adjacent to the base is 40° . Find, geometrically, the locus of the vertex of the triangle. N.B.—A locus found by trial is not sufficient.

Solution by Professor A. DROZ-FARNY, and L. E. REAY, B.A.

Soit BC la base fixe du triangle ABC ; la différence des angles à la base étant constante, les faisceaux décrits par les côtés BA et CA sont égaux et de sens de rotation contraire ; il en résulte d'après un théorème bien connu (Voir CASEY, *Analytical Geometry*, second edition, p. 289) que le lieu des sommets est une hyperbole équilatère, de diamètre BC et dont les asymptotes ont la direction des bissectrices de l'angle A.

13499. (Professor W. BOOTH.)—If $\alpha, \beta, \gamma, \delta$ be the roots of

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

 express $4(\beta\gamma - \alpha\delta) [(a + \delta)\beta\gamma - \alpha\delta(\beta + \gamma)]$

$$-(\beta + \gamma - \alpha - \delta)^2 [(a + \delta)\beta\gamma + \alpha\delta(\beta + \gamma)]$$

 in terms of the coefficients.

Solution by L. E. REAY, B.A.; Rev. R. HARLEY, M.A., F.R.S.; and others.

$$\begin{aligned} & (\beta\gamma - \alpha\delta) [(a + \delta)\beta\gamma - \alpha\delta(\beta + \gamma)] - (\beta + \gamma - \alpha - \delta)^2 [(a + \delta)\beta\gamma + \alpha\delta(\beta + \gamma)] \\ &= \beta\gamma(a + \delta)(2\Sigma\alpha\beta - \Sigma\alpha^2) - 8\alpha\beta\gamma\delta(a + \delta) \\ & \quad + \alpha\delta(\beta + \gamma)(2\Sigma\beta\gamma - \Sigma\alpha^2) - 8\alpha\beta\gamma\delta(\beta + \gamma) \\ &= (2\Sigma\alpha\beta - \Sigma\alpha^2)\Sigma\beta\gamma\delta - 8\alpha\beta\gamma\delta\Sigma\alpha = [4\Sigma\alpha\beta - (\Sigma\alpha)^2]\Sigma\beta\gamma\delta - 8\alpha\beta\gamma\delta\Sigma\alpha \\ &= \left(\frac{24c}{a} - \frac{16b^2}{a^2}\right)\left(-\frac{4d}{a}\right) + \frac{8e}{a} \frac{4b}{a} = \frac{32}{a^3}(2b^2d - 3acd + abe). \end{aligned}$$

4653. (MATTHEW COLLINS, LL.B.)—Let AC be the side of a regular pentagon inscribed in a circle whose centre is D ; on AO draw an equilateral triangle AOB ; bisect the arc BC in D, and the arc AD in E ; then prove that the chord AE will be very nearly equal to the side of a regular undecagon inscribed in the circle.

Solution by H. W. CURJEL, M.A.

Let c = length of circumference of circle.
 Arc AC = $\frac{1}{5}c$, arc AB = $\frac{1}{3}c$; $\therefore$ arc AD = $\frac{1}{2}$ (arc AC + arc AB) = $\frac{1}{3}\frac{1}{5}c$;
 therefore arc AE = $\frac{1}{3}\frac{1}{5}c = \frac{1}{15}c$ very nearly ;
 therefore chord AE is very nearly equal to a side of a regular undecagon inscribed in the circle.

3804. (Professor CROFTON, F.R.S.)—If five forces act along the sides of any convex pentagon, show that they cannot make equilibrium

unless three act one way and two the opposite way; and that the two latter must not act along two adjacent sides of the pentagon.

Solution by H. J. WOODALL, A.R.C.S.

For equilibrium the resultants of two pairs of sides must (1) meet on the odd side; (2) must have resultant along line of odd side; (3) must have resultant in opposite direction to force on odd side.

(a) Suppose all the forces act in the same (clockwise) sense; then it is impossible to satisfy (1).

(b) Let one force act in anti-clockwise sense; we can satisfy (1), (2), but not (3).

(c) If two forces act in anti-clockwise sense.—(i.) If these forces are in adjacent sides, we can satisfy (1) and (2), but not (3); (ii.) if these forces are in non-adjacent sides, we can satisfy (1), (2), and (3). The figures can be easily supplied.

13517. (H. MACCOLL, B.A.)—Dividing all statements into three classes—(e) *certainties* (always true), (η) *impossibilities* (never true), (θ) *variables* (possibly true, possibly false)—to which classes must the statements α , β , u , v respectively belong, so that the implication

$$(u_\alpha + u_\beta)(\alpha_\alpha + \beta_\alpha) : u_\alpha$$

(denoted by ϕ) may be a *certainty*? To which classes so that ϕ may be an *impossibility*? *Defn.*—The three symbols AB, A + B, A_β (or A : B) respectively assert that A and B are *both* true; that either A or B is true; that if A is true B is true. *Note.*—There are several solutions, of which one is that ϕ is an *impossibility* when α and u are both *certainties*, and β and v both *impossibilities*.

Solution by the PROPOSER.

By the methods given in my Fifth Paper "On the Calculus of Equivalent Statements" (see *Proc. Lond. Math. Soc.*), we readily get

$$\phi! u^\alpha v^\alpha + \alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha + u^\alpha v^\alpha (\alpha^\alpha + \beta^\alpha) + u^\alpha v^\alpha (\alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha) + u^\alpha v^\alpha (\alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha),$$

and

$$\phi! (u^\alpha v^\alpha + u^\alpha v^\alpha)(\alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha).$$

All other combinations of $u^\alpha v^\alpha$, $u^\alpha v^\alpha$, $\alpha^\alpha \beta^\alpha$, $\beta^\alpha \gamma^\alpha$, not found among the above terms, nor multiples of them, are combinations from which we can neither infer ϕ nor ϕ ; so that the above is a complete solution of the Question on the given conditions.

Note.—The solution is simplified by first reducing the given implication ϕ to the form $u_\alpha \beta_\alpha + u_\beta \alpha_\beta : u_\alpha$; a reduction easily effected. The working may then be abbreviated by taking account at starting of the self-evident inductive (or inverse) implications $\phi! u^\alpha + v^\alpha$ and $\phi! \alpha_\beta \beta_\alpha! \alpha^\alpha \beta^\alpha + \alpha^\alpha \beta^\alpha$. The symbol A'' means (A') , and A''' means (A'') ; that is, A'' means $A' + A'$, and A''' means $A' + A'$. The symbol $A! B$ means $B : A$.

4749. (A. F. TORRY.)—From a point, at distance a from a centre of force varying as $r^{-2} + 2ar^{-3}$, a particle is projected at an inclination of $\frac{1}{2}\pi$ to the initial distance. Determine the different orbits described, according as the initial velocity bears to that in a circle at the same distance a ratio greater than, equal to, or less than $2 : \sqrt{3}$.

Solution by H. W. CURJEL, M.A.

Let mP be the force at distance, where m is the mass of the particle, and let x = the ratio of the initial velocity to that in a circle of radius a .

$$\text{Orbit is given by } \frac{d^2u}{d\theta^2} + u = \frac{u^2 + 2au^3}{h^2u^2} \frac{a^2P}{3},$$

$$\text{where } h^2 = \frac{x^2 a^2 P}{2} = \frac{2(1+2au)}{3x^2 a};$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{4(u+au^2)}{3x^2 a} - u^2 + k = \frac{4u+au^2}{3x^2 a} - u^2 + \frac{2}{a^2} - \frac{8}{3x^2 a^2},$$

for $(du/d\theta)^2 = 1/a^2$ when $u = 1/a$;

$$\therefore \frac{d\theta}{x\sqrt{3}} = \frac{adu}{\{(4-3x^2)a^2u^2 + 4au - 2(4-3x^2)\}^{\frac{1}{2}}}.$$

If $x = 2/\sqrt{3}$, $\frac{1}{2}\theta = (a/r)^{\frac{1}{2}} - 1$, if $\theta = 0$ when $r = a$.

If $x > 2/\sqrt{3}$,

$$\frac{\theta}{x\sqrt{3}} = \frac{1}{(3x^2-4)^{\frac{1}{2}}} \left\{ \sin^{-1} \frac{(3x^2-4)(a/r)-2}{(18x^4-48x^2+36)^{\frac{1}{2}}} - \sin^{-1} \frac{3x^2-6}{(18x^4-48x^2+36)^{\frac{1}{2}}} \right\}.$$

$$\text{If } x < 2/\sqrt{3}, \quad \frac{\theta}{2\sqrt{3}} = \frac{1}{(4-3x^2)^{\frac{1}{2}}}$$

$$\log \frac{(a/r)(4-3x^2) + 2 + \{(4-3x^2)^2(a^2/r^2) + 4(a/r)(4-3x^2) - 2(4-3x^2)^2\}^{\frac{1}{2}}}{6-3x^2 + \{(4-3x^2)3x^2\}^{\frac{1}{2}}}.$$

13451. (R. KNOWLES, B.A.)—From a variable point T on the tangent at the vertex A of the parabola $y^2 = 4ax$, tangents TA , TP are drawn; the circle PTA cuts the parabola given in CD ; the diagonals CA , PD intersect in G . Prove that the equation to the locus of G is $ay^2 = 2x^2(x-a)$.

Solution by Rev. J. CULLEN, B.A.; and LIONEL E. REAY, B.A.

Let P be $\{4am^2, 4am\}$; then T is $\{0, 2am\}$, and the circle through these points and the origin is

$$x^2 + y^2 - 2a(1+2m^2)x - 2amy = 0.$$

Putting $x = 4a\mu^2$, $y = 4a\mu$, we see that this circle intersects the parabola in two other points $\{4a\mu_1^2, 4a\mu_1\}$ and $\{4a\mu_2^2, 4a\mu_2\}$, where μ_1 and μ_2 are the roots of

$$2\mu^2 + 2m\mu + 1 = 0.$$

The equations of CA and PD are respectively

$$y\mu_1 = x \quad \text{and} \quad x - (m + \mu_2)y + 4a\mu_2 = 0.$$

Substituting the values of μ_1 and μ_2 in terms of m and then eliminating m , we get $ay^2 = 2x^2(x - a)$.

4816. (Professor EVANS, M.A.)—A. hits a circular target of twenty inches radius, at the distance of four hundred yards, five hundred times out of one thousand shots with a rifle. How many times out of one thousand shots with the rifle must B. hit the same target, at the same distance, to show that his skill is to the skill of A. as two to one?

Solution by K. S. PUTNAM.

It would seem that in comparing the skill of A. and B. from the data we should compare A.'s ratio of hits and misses with B.'s ratio. If A. hits the same number of times he misses, B., to be twice as skilful, should hit twice to missing once, and should hit the target $666\frac{2}{3}$ times out of a thousand shots; or, out of 150 shots, A. hits the target 75 times, and B. 100 times.

[Mr. WHITWORTH, having seen the above solution of this question, remarks as follows:—"The question turns on what is meant by B.'s skill being double of A.'s. Mr. PUTNAM's interpretation is quite arbitrary, and it seems quite unreasonable.

Reasonably one shooter can only be said to be twice as skilful as another when the chance of the first hitting the target is double the chance of the second. But on this interpretation the question requires that B.'s skill be infinite.

If all that we know of a shooter is that in m trials he has hit the target n times, the expectation that he succeed the next time is, on the arbitrary hypothesis that all degrees of skill are equally likely,

$$\begin{aligned} \frac{\int_0^1 x^{n+1} (1-x)^{m-n} dx}{\int_0^1 x^n (1-x)^{m-n} dx} &= \frac{B(n+2, m-n+1)}{B(n+1, m-n+1)} \\ &= \frac{\Gamma(n+2) \Gamma(m+2)}{\Gamma(n+1) \Gamma(m+3)} = \frac{(n+1)! (m+1)!}{n! (m+2)!} = \frac{n+1}{m+2}. \end{aligned}$$

Therefore, if A. has been observed to hit the target a times in 1000 trials, and B. to hit it b times, B.'s skill is double A.'s provided $b+1 = 2(a+1)$ or $b = 2a+1$.

In the question, substitute 499 for 500 and the answer will be 999.”]

Note by Rev. T. C. SIMMONS, M.A.

The two solutions given of Question 4816 (see above), though interesting, are fundamentally defective. In the first place, they presume the shots to fall, like hailstones, equally on all parts of the target, which in practice never happens. Secondly, they make no allusion to previously employed scientific methods of estimating the skill of a marksman, of which the three following are the best:—(1) His skill varies inversely as his “mean error,” i.e., the average distance of his points of impact from the centre; (2) it varies inversely as his “error of mean square,” i.e., the square root of the mean of the squares of those distances from the centre; (3) it varies inversely as his “probable error,” i.e., the radius of the concentric circle containing half of the points of impact of the total number discharged. Of these three methods, the first is the simplest in practice; and the second, as LIAGRE remarks, is scientifically the most accurate. But when the number of shots is very large, the “mean error,” the error of “mean square,” and the “probable error” will all bear a fixed ratio to one another; so that in solving Quest. 4816 we may accurately employ the third method, elaborately developed by Sir JOHN HERSCHEL on pp. 495–507 of his *Familiar Lectures on Scientific Subjects* (Strahan, 1866).

Attempting the subject on a previous occasion, HERSCHEL went entirely wrong (see pp. 398–401 of his *Essays from the Edinburgh and Quarterly Reviews*, &c., Longmans, 1857), as he had based his calculations on the assumption that, for a given marksman, the chance of any single shot landing between distances r and $r + dr$ from the centre of the target is proportional to $e^{-k^2 r^2} dr$. In the *Familiar Lectures* he corrected himself, practical observation having shown him in the meanwhile what is theoretically evident when once pointed out, namely, that the said chance ought instead to be proportional to $e^{-k^2 r} r dr$. And by integration he deduced the following singularly neat formula, whose strict applicability to practice he illustrated by elaborate tabulations of the published results of upwards of 474,000 shots at circular targets at the Grand National Archery Meetings for 15 successive years, commencing with 1850. Let a be the radius of the circle within which, out of a very large number of shots, a man lands half his total; then his chance of missing, in any single shot, a concentric circle of radius r is $(\frac{1}{2})^{r^2/a^2}$.

By means of this formula (which holds true whatever system we adopt for measuring skill) the solution of Quest. 4816 is easy. A. lands 500 out of 1000 shots within a distance of 20 inches from the centre. If we define (as HERSCHEL does) that B., to be twice as skilful, must land 500 out of 1000 shots within a distance of 10 inches from the centre, his chance of landing a given shot further than 20 inches from the centre is $(\frac{1}{2})^4$ or .0625. Hence, on the average, out of every thousand shots he would hit the target 937½ times, which is the required answer. In the same manner, a third marksman, C., whose “probable error” was 6½ inches, would hit 998 out of 1000 times; and a fourth D., whose “probable error” was 40 inches, would hit 169 out of 1000 times. According to HERSCHEL’s definition, C. would in such case be three times as skilful as A., and D. half as skilful as A.

It has, of course, been presumed that the points of impact are

symmetrically distributed all round the target. This will not be likely to happen when, for instance, a strong wind is blowing crosswise. If the wind is fitful, there will probably be no symmetry of any kind, and in such case the accurate solution of Quest. 4816 is impossible. If the wind blows quite steadily (which might horizontally have the same kind of influence as the inaccurate sighting of a gun might have vertically), there might possibly be more or less symmetric distribution round a centre not coinciding with the centre of the target. Or it might happen in any given case that the tendency to "error" in one direction was uniformly greater than the tendency to error in the perpendicular direction. On this matter HERSCHEL has nothing to say, and it is best to consult some of the French treatises. LIAGRE deals with it on pp. 321-335 of his *Calcul des Probabilités* (Brussels, 1879), and BERTRAND in his *Calcul des Probabilités* (Paris, 1889), pp. 232-246. Both of these writers, by independent analysis, find the curve of equal distribution to be in general an ellipse, and for the particular case of a circle they both allow the chance of a shot falling between distances r and $r + dr$ from the centre to be proportional to $e^{-k^2 r^2} r dr$. But, in a book recently published, *Théorie des Jeux de Hasard* (Gauthier-Villars, Paris), by H. LAURENT, the author states the said chance to vary as $e^{-k^2 r} dr$, thus falling once more into HERSCHEL's original mistake of 1857. M. LAURENT is a well-known and eminent authority on Probability, but he gravely applies this incorrect formula (pp. 113, 115) to the solution of an interesting problem about the chance of hitting a circular target which rapidly revolves about a vertical axis. HERSCHEL's article apparently is as little remembered in France as in England. Unfortunately the *Familiar Lectures* seem to be out of print, and difficult to obtain. Perhaps the present mention may help to rescue them from their comparative and quite undeserved oblivion.

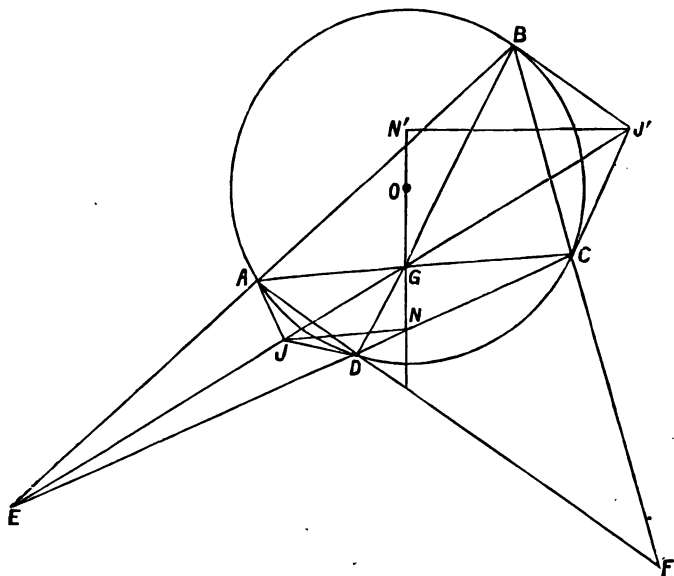
To conclude, it may be asked, perhaps, why the skill of a marksman should not be considered to vary inversely as the area, rather than the radius, of the circle which receives half of his total number? There is a very plain answer, not given by HERSCHEL, which may be reserved for a future occasion. But, whether the area or the radius be employed as basis, the relative skill of A. compared with B. comes out the same for all-sized circular targets. But both Mr. PUTNAM's and Mr. WHITWORTH's systems would make the relative skills to depend upon the size of the target. For instance, if k denote the proportion of successes to failures, and A.'s $k = 1$, B.'s $k = 2$ for a target of 20-inch radius, it is easily found that A.'s k would = 15 and B.'s $k = 80$ for a 40-inch target: so that, whereas B.'s skill, according to Mr. PUTNAM, is double of A.'s for the smaller target, it is $5\frac{1}{2}$ times A.'s skill for the larger one, which is absurd. A similar objection applies to Mr. WHITWORTH's proposed system, which also would make the relative skill of B. versus A. to increase with the size of the target. Moreover, it is difficult to accept the latter's statement that a man who succeeds every time must needs be of infinite skill. For A.'s "probable error" being 20 inches, a man D., whose "probable error" was 5 inches, would hit 65535 out of every 65536 times, and might fairly be considered, in the case of the first 1000 trials, to succeed every time. But, in such case, D.'s skill compared with A.'s certainly cannot be reckoned as infinite. Infinite skill, to my mind, would rather be that of the man who planted every one of his 1000 shots exactly in the centre of the target, if such a thing were possible.

13547. (R. KNOWLES, B.A.)—ABCD is a quadrilateral inscribed in a circle centre O; the diagonals AC and BD intersect in G; TN, T'N' are the perpendiculars from T, T', the poles of AD and BC on GO; prove that

$$TN : T'N' = TG : T'G.$$

Solution by W. E. JEFFARES, B.A. ; and Professor A. DROZ-FARNY.

Produce BA and CD to meet in E, and AD and BC in F. Since BE and AF are chords drawn from F to cut the circle in four points, the joins of these points directly and transversally (viz., the lines CD and AB, also BD and AC) intersect on the polar of F (see McDOWELL'S *Exercises in Modern Geometry*, p. 258, Ex. 241), in E and G. Now, if



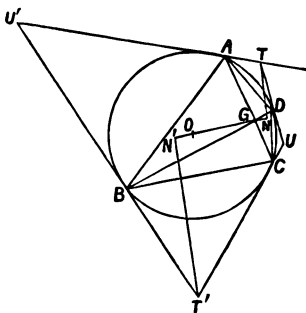
tangents are drawn from A and D, they meet in T on the polar of F (see solution of Quest. 13204 in Vol. LXVI., p. 57). Also, if tangents are at B and C, they meet on the poles of F; therefore E, T, G, T' are collinear. Now, if perpendiculars are drawn from T and T' on the line OG, the triangles T'GN and T'GN' are equiangular, and therefore

$$TN : T'N' :: TG : T'G.$$

[Professor SANJANA solves the Question as follows:—

As ABCD is an inscribed and TUT'U' a circumscribed quadrilateral, therefore TT' is concurrent with AC and BD.

Hence TGT' is a straight line, and the similar triangles TGN, T'GN' give the required proportion.]



13522. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Find the product of the series— $1 + 3 + 6 + 10 + 15 + 21 + \dots$ to n terms, by the series— $1 + 6 + 18 + 40 + 75 + 126 + \dots$ to n terms; and show that n , $(n+1)$, and $(n+2)$ are factors of the product.

Solution by the Rev. ROBERT HARLEY, M.A., F.R.S.; J. WOODCOTE WEST; and many others.

Let S_1 represent the sum of the first series, and S_2 that of the second, to n terms.

$S_1 = \frac{1}{2}n(n+1)(n+2)$, as is well known.

To find S_2 : The $(n+1)$ th term of the second series is $\frac{1}{2}(n+1)^2(n+2)$, which, adopting the factorial notation

$$n(n-1)(n-2) \dots (n-m+1) = n^{(m)},$$

may be written in the form $\frac{1}{2}n^{(3)} + \frac{1}{2}n^{(2)} + 5n + 1$.

Integrating this expression, and determining the arbitrary constant (which in this case is zero) by a reference to the first term of the series,

we obtain $S_2 = \frac{1}{2}n^{(4)} + \frac{7}{2}n^{(3)} + \frac{5}{2}n^{(2)} + n$,

or, more simply, $S_2 = \frac{1}{2}n(n+1)(n+2)(3n+1)$;

therefore $S_1 S_2 = \frac{1}{4}n^2(n+1)^2(n+2)^2(3n+1)$,

and both S_1 and S_2 , as well as their product, contain the factors n , $n+1$, and $n+2$.

13573. (L. E. REAY, B.A. Suggested by Quest. 13540.)—A point B moves on a fixed conic. A and D are fixed points on the conic; BC passes through D; the angle BAC is constant. Find the locus of C.

Solution by Rev. J. CULLEN, B.A.

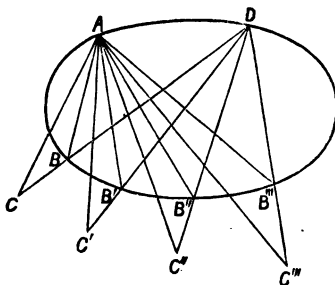
The anharmonic pencil

$$\begin{aligned}\{A.BB'B''B'''\} \\ &= \{D.BB'B''B'''\} \\ &= \{D.CC'C''C'''\},\end{aligned}$$

also, since the angles are equal,

$$\begin{aligned}\{A.BB'B''B'''\} \\ &= \{A.CC'C''C'''\} \\ &= \{D.CC'C''C'''\}.\end{aligned}$$

Therefore the locus of C is a conic also passing through A and D.



13557. (Rev. T. WOOD ROBINSON, R.N.)—Find the chance of throwing all sixes in r throws with n dice, and deduce the number of throws that must be made with five dice that the chance of throwing all sixes may be even.

Solution by the PROPOSER.

$$\begin{aligned}\text{Chance of throwing 1 six in 1 throw with } n \text{ dice } & n/1 \cdot 5^{n-1}/6^n, \\ \text{,, ,, 2 ,, 1 ,, ,, } & [n(n-1)]/(1 \cdot 2) 5^{n-2}/6^n, \\ \text{,, ,, , } r \text{ ,, 1 ,, ,, } & [n(n-1) \dots (n-r+1)]/(1 \cdot 2 \dots r) 5^{n-r}/6^n.\end{aligned}$$

To throw all sixes in 2 throws with n dice.

We may throw 0 and n , chance $5^n/6^n \cdot 1/6^n$;

$$\text{or ,, 1 and } n-1, \text{ ,, } n/1 \cdot 5^{n-1}/6^n \cdot 1/6^{n-1},$$

$$\begin{aligned}\text{Total chance} &= 1/6^n \left\{ \left(\frac{5}{6}\right)^n + n/1 \cdot \left(\frac{5}{6}\right)^{n-1} + \dots \right\} = 1/6^n \left\{ 1 + \frac{5}{6} \right\}^n \\ &= \left\{ 1 - \left(\frac{1}{6}\right)^2 \right\}^n.\end{aligned}$$

To throw all sixes in 3 throws with n dice.

We may throw 0 and then n in two throws, chance $= 5^n/6^n \left\{ 1 - \left(\frac{1}{6}\right)^2 \right\}^n$.

$$\text{or 1 and then } n-1 \text{ in two throws, chance} = n/1 \cdot 5^{n-1}/6^n \left\{ 1 - \left(\frac{1}{6}\right)^2 \right\}^{n-1},$$

$$\begin{aligned}\text{Therefore total chance} &= 1/6^n \left[5 \left\{ 5 - \left(\frac{1}{6}\right)^2 \right\}^n 5^n + n \left\{ \square \right\}^{n-1} 5^{n-1} + \dots \right] \\ &= 1/6^n \left[1 + 5 \left\{ 1 - \left(\frac{1}{6}\right)^2 \right\} \right]^n \\ &= \left[1 - \left(\frac{1}{6}\right)^3 \right]^n.\end{aligned}$$

Therefore chance of throwing all sixes in r throws with n dice is

$$\left[1 - \left(\frac{1}{6}\right)^r \right]^n.$$

angles XMA , $CA'F'$ are equivalent; therefore $AM = CA'$; therefore the locus of A' is a cissoid with cusp at A and CD as asymptote.

(c) FX' makes the same angle with $X'F'$ that $F'X$ makes with XF , i.e., a right angle. Therefore FX' touches a circle of radius $2a$ as the circle rolls along LM , parallel to $F'X$, where $MD = a$. The locus of X' is therefore the curve $r = 2a \cot \frac{1}{2}\theta$ referred to F as origin and the line through F parallel to the directrix as initial line. The curve has LM as asymptote and a loop as in figure.

13437. (Professor BERNÉS.)—Par le sommet de l'angle A du triangle ABC on mène deux isogonales quelconques Ax , Ay , et, par le milieu D de BC , on trace, entre ces deux droites, le segment MN partagé en D en deux parties égales. Les droites BN , CN rencontrant AM en β , γ , démontrer que les tangentes en B et C aux circonférences $AB\beta$, $AC\gamma$ se rencontrent sur AN .

Solution by H. W. CURJEL, M.A.

Let the tangents at C and B to circles $AC\gamma$, $AB\beta$ cut AN in H , H' , and let $C\gamma$ cut AB in K .

Then $\angle HCK = \angle CA\gamma = \angle KAH$;

$\therefore \angle CHA = \angle CKA = \angle MBA$,

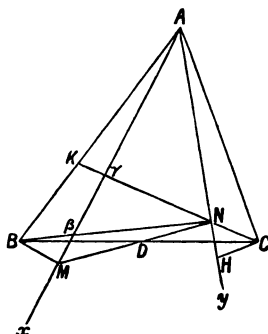
for BM is parallel to CN .

$\therefore AH : AC = AB : AM$.

Similarly,

$AH' : AB = AC : AM$;

therefore H and H' coincide.



11857. (Professor TISSOT.)—Du centre d'un cercle intérieur et bitangent à une conique, on abaisse une perpendiculaire sur une tangente à la courbe, et l'on considère les deux points de cette perpendiculaire qui se trouvent à une distance de son pied égale à la longueur de la tangente menée de ce pied au cercle. Trouver le lieu décrit par l'un de ces points, (1) lorsque la tangente est fixe et que le cercle bitangente varie, (2) lorsque, le cercle restant le même, le point de contact de la tangente se déplace sur la conique.

Solution by H. J. WOODALL, A.R.C.S.

Let the centre (H) of circle be (hk) , the equation to the conic be $x^2/a^2 + y^2/b^2 = 1$, and P be (θ) .

Distance $HP = \{(h - a \cos \theta)^2 + (k - b \sin \theta)^2\}^{\frac{1}{2}}$.

Tangent at P is $x \cos \theta / a + y \sin \theta / b = 1$.

Length of perpendicular from H on this tangent

$$= \left(\frac{h}{a} \cos \theta + \frac{k}{b} \sin \theta - 1 \right) / \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)^{\frac{1}{2}}.$$

“Length of tangent”

$$= \left[\{ (h - a \cos \theta)^2 + (k - b \sin \theta)^2 \} (\cos^2 \theta / a^2 + \sin^2 \theta / b^2) - (h \cos \theta / a + k \sin \theta / b - 1)^2 \right]^{\frac{1}{2}} / (\cos^2 \theta / a^2 + \sin^2 \theta / b^2)^{\frac{1}{2}}.$$

Points whose locus is required are the intersections of

$$(x - h) a \sin \theta = (y - k) b \cos \theta \dots \dots \dots (1),$$

and $b x \cos \theta + a y \sin \theta - ab$

$$= \pm \left[\{ (h - a \cos \theta)^2 + (k - b \sin \theta)^2 \} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) - (b h \cos \theta + a k \sin \theta - ab)^2 \right]^{\frac{1}{2}} \dots (2),$$

(1) when (h, k) are connected by the equation to the evolute, i.e.,

$$(ah)^{\frac{2}{3}} + (bk)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}};$$

(2) when θ varies and must be eliminated between these equations.

But (1) it can be easily shown that the required locus is a pair of straight lines inclined at an angle of 45° to the given tangent. The equation is, therefore,

$$bx \cos \theta + ay \sin \theta - ab = \pm \{ ax \sin \theta - by \cos \theta - (a^2 - b^2) \sin \theta \cos \theta \},$$

as may be proved easily (!) by analysis.

Equation (2), above, reduces to

$$\pm (bx \cos \theta + ay \sin \theta - ab) = ax \sin \theta - by \cos \theta - (a^2 - b^2) \sin \theta \cos \theta.$$

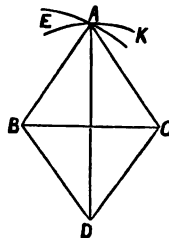
By eliminating θ between this and (1), we get

$$a^4 (x - h)^2 + a^2 b^2 (y - k)^2 = [a (y - k)(y - h) + a (x + k)(x - h) + (a^2 - b^2)(y - k) \{ 1 + b^2/a^2 \{ (y - k)/(x - h) \}^2 \}]^2.$$

13592. (R. CHARTRES.)—Given two sides of a triangle ABC, and Σ (FA), construct the triangle, F being FERMAT'S point, so that Σ (FA) is a minimum.

Solution by the PROPOSER.

Let BC be one of the given sides. Describe an equilateral triangle BCD on it. With D as centre and radius = Σ (FA) describe the arc EAK. With centre B and radius = the other given side, describe a circle cutting the other circle in A. Then evidently ABC is the required triangle.



13565. (I. ARNOLD.)—The ends of a cord, 10 feet long, are fastened to two points, A and B, in the same horizontal line and 6 feet apart. From the cord are suspended two weights, one of 3 lbs., the other 5 lbs. The portion of the cord between the points of suspension is also horizontal, and 3 feet perpendicular distance from AB. It is required to find the lengths of those portions of the cord between A and B and the points of suspension of the weights.

Solution by the Rev. J. CULLEN, B.A.

Let T be the tension of the horizontal portion of the string.

$$T = 3 \cot \theta = 5 \cot \phi,$$

$$u = 3 \cot \theta, \quad v = 5 \cot \phi.$$

Therefore $u/v = 5/3$.

$$\text{Also } 6 - (u + v) = 10 - (x + y),$$

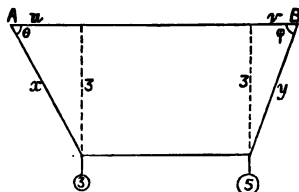
$$\text{and } x = (9 + u^2)^{\frac{1}{2}}, \quad y = (9 + v^2)^{\frac{1}{2}}.$$

Hence we have

$$20v^3 + 43v^2 - 12v - 45 = 0,$$

There is only one real root = .953 ;

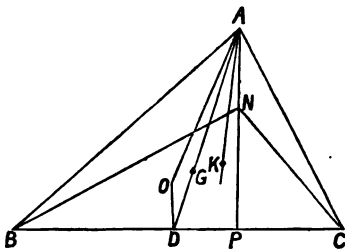
therefore $y = 3.147$ ft., $x = 3.393$ ft.



11339. (Professor LEMOINE.)—L'orthocentre n'est jamais sur le cercle de BROCARD, sauf dans le cas limite qui correspond au triangle équilatéral.

Solution by H. W. CURJEL, M.A.

Since in any triangle the symmedian point K, and therefore the BROCARD circle, lies within the circumcircle, it is obvious that in the case of an obtuse-angled or right-angled triangle the orthocentre N lies without the BROCARD circle. Also, in an isosceles triangle, excluding of course the case of the equilateral triangle, in which N coincides with the circumcentre O and K and therefore with the BROCARD circle, which reduces to a point, N is collinear with O, K, and the centroid G; and G, and therefore K, lies between O and N; therefore N lies without the BROCARD circle; therefore the scalene acute-angled triangle alone remains to be considered. Let ABC be such a triangle, O lying within the $\angle BNC$ (it clearly cannot lie on AN, BN, or CN). Let AN meet



BC in P. Then, if $B < C$, O is within the $\angle BAP$, and G, and therefore also K, lies within $\angle OAP$; therefore both O and K lie within $\angle BNP$, which is acute ($= O$); therefore ON cuts the BROCARD circle between O and N, and therefore N is without the BROCARD circle in all cases except that of the equilateral triangle, and clearly also cannot be on it.

13032. (M. BRIERLEY.)—Two given circles, of centres O, P, intersect in A, B; required the length of the common chord AB when the ratio of the sum of the two lunettes O, P to the intercepted position of the two circles is as m to n .

Solution by REV. J. L. KITCHIN, M.A.

Let $AD = 2AH = 2k$,
 $\phi = \sin^{-1} k/r_1$, $\theta = \sin^{-1} k/r_2$,
 r_1, r_2 be the radii of the circles.

Sector $APD = r_2^2 \theta$,
 therefore segment

$ACD = r_2^2 \sin^{-1} k/r_2 - k(r_2^2 - k^2)^{\frac{1}{2}}$,
 and

$ABD = r_1^2 \sin^{-1} k/r_1 - k(r_1^2 - k^2)^{\frac{1}{2}}$.

$$\frac{2 \text{ lunettes}}{2 \text{ segments}} = \frac{\pi(r_1^2 + r_2^2) - 2 \text{ segments}}{2 \text{ segments}} = \frac{m}{n},$$

$$\pi(r_1^2 + r_2^2) = (m + n/n) 2 \text{ segments};$$

therefore

$$[\pi n(r_1^2 + r_2^2)]/(m + n) = r_1^2 \sin^{-1} k/r_1 + r_2^2 \sin^{-1} k/r_2 - kr_1(1 - k^2/r_1^2)^{\frac{1}{2}} - kr_2(1 - k^2/r_2^2)^{\frac{1}{2}}.$$

Expanding the dexter, and neglecting higher powers of k than the third, this becomes $= \frac{2}{3}(1/r_1 + 1/r_2) k^3$; therefore

$$(2k)^3 = \frac{12n\pi(r_1^2 + r_2^2)r_1r_2}{(r_1 + r_2)(m + n)}; \quad \therefore AD = \left\{ \frac{12n\pi(r_1^2 + r_2^2)r_1r_2}{(r_1 + r_2)(m + n)} \right\}^{\frac{1}{3}}.$$

13582. (EDITOR.)—The axes of an ellipse are 10 and 20; find the mean length of all the radius-vectors drawn from (1) the centre, (2) the focus.

Solution by E. P. BARRETT, B.A.

(1) The mean value M

$$= \frac{1}{a-b} \int_0^a r dr = \frac{1}{a-b} \frac{a^2 - b^2}{2} = \frac{a+b}{2} = 7\frac{1}{2}.$$

(2) In this case M

$$= \frac{1}{2ae} \int_{a(1-e)}^{a(1+e)} r dr = \frac{1}{2ae} \frac{a^2(1+e)^2 - a^2(1-e)^2}{2} = a = 10.$$

[Mr. LIONEL E. REAY solves the Question as follows :—

$$e^2 = 1 - b^2/a^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

(1) Referred to centre, ellipse is

$$r^2 = \frac{25}{1 - \frac{3}{4} \cos^2 \theta}; \quad \therefore r = \frac{5}{(1 - \frac{3}{4} \cos^2 \theta)^{\frac{1}{2}}};$$

therefore mean value of $r = \frac{1}{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{5d\theta}{(1 - \frac{3}{4} \cos^2 \theta)^{\frac{1}{2}}},$

which can be expressed as the following series :—

$$5 \left\{ 1 + \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\frac{3}{4}\right)^3 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\frac{3}{4}\right)^4 + \dots \right\}.$$

(2) Referred to focus, ellipse is $r = \frac{5}{1 - (\frac{3}{4})^{\frac{1}{2}} \cos \theta};$

therefore mean value

$$\begin{aligned} &= \frac{1}{\pi} \int_0^{\pi} \frac{5d\theta}{1 - (\frac{3}{4})^{\frac{1}{2}} \cos \theta} = \frac{5}{\pi} \frac{2}{\sqrt{\frac{1}{4}}} \left\{ \tan^{-1} \left(\frac{1 + (\frac{3}{4})^{\frac{1}{2}}}{1 - (\frac{3}{4})^{\frac{1}{2}}} \right)^{\frac{1}{2}} \tan \frac{1}{2}\theta \right\}_0^{\pi} \\ &= \frac{5}{\pi} \cdot 4 \cdot \frac{1}{2}\pi = 10. \end{aligned}$$

13507. (L. E. REAY, B.A.)—An ellipse is cut at P and P' by a tangent to its minor auxiliary circle. Prove $PP' = HP \sim SP'$.

Solution by the PROPOSER.

Let PP' cut the director circle in Q, Q'.

$$QR^2 = CQ^2 - CR^2 = a^2 + b^2 - b^2 = a^2;$$

$$\therefore QR = a;$$

$$\therefore QQ' = 2a = QP + PQ';$$

$$\therefore QP + PQ' = SP + HP.$$

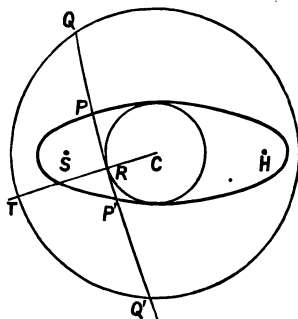
$$\begin{aligned} QP \cdot PQ' &= QR^2 - PR^2 \\ &= CQ^2 - CR^2 - PR^2 \\ &= a^2 + b^2 - CP^2 \\ &= CD^2 = SP \cdot PH. \end{aligned}$$

$$\therefore QP = SP \text{ (or HP),}$$

$$\text{and } PQ' = HP \text{ (or SP).}$$

$$\text{Similarly } Q'P' = SP'.$$

$$\text{Hence } PP' = HP \sim SP' = HP' \sim SP.$$



13477. (The late Rev. W. ROBERTS, M.A.)—Find the parallel to $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$, which is FRESNEL'S "surface of elasticity"; it is also the pedal of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 - 1 = 0$.

Solution by Rev. J. CULLEN, B.A.

The envelope of the sphere $2\Sigma ax = \Sigma a^2 - D$ (1),

where $D = k^2 - \Sigma x^2$, $k = \text{constant}$, $\Sigma a^2 a^2 = (\Sigma a^2)^2$ (2),

is the required parallel surface.

Differentiating (1) and (2), we have

$$\Sigma (a^3 a - 2a \Sigma a^2) da = 0, \quad \Sigma (x - a) da = 0;$$

$$\therefore a^3 a - 2a \Sigma a^2 = \lambda (x - a), \quad \&c.$$

Multiplying by a , &c., adding, and employing (1) and (2) to determine λ , and putting $t = 2D\Sigma a^2/(\Sigma a^2 + D)$, we easily find

$$\alpha = \frac{t^2}{2D-t} \cdot \frac{x}{a^2-t}, \quad \beta = \frac{t^2}{2D-t} \cdot \frac{y}{b^2-t}, \quad \gamma = \frac{t^2}{2D-t} \cdot \frac{z}{c^2-t}.$$

Whence, substituting in (1) and (2), we see that the surface is obtained by eliminating t between the equations

$$\Sigma x^2/(a^2-t) = D(t-D)/t^2 \quad \text{and} \quad \Sigma a^2 x^2/(a^2-t)^2 = D^2/t^2,$$

and may therefore be expressed as a determinant by the dialytic method.

Since the "surface of elasticity" is the inverse of the reciprocal of the ellipsoid, it is also the pedal.

11193. (G. F. HOWSE.)—If the moments of a system of forces acting on a rigid body vanish about five sides of two triangles not in the same plane, prove that they vanish about the sixth also.

Solution by the PROPOSER.

The forces are clearly equivalent either to a single resultant in the plane of a third triangle or a couple parallel to that plane. But, if they are equivalent to such a single resultant, its line of action must meet two sides of the second triangle and one of the first, which is impossible. And, if they are equivalent to such a couple, it must be parallel to the second triangle, which is also impossible. Hence the forces are at rest.

11122. (E. B. ELLIOTT, M.A.)—S, U, V, W are four fixed quadrics having a common plane section; through any point P are drawn six other quadrics having the same common plane section, and each passing also

through the second plane section common to some two of S, U, V, W . Show that these six have one more point common. Show also that, for different positions of P lying in any fixed plane, the locus of this second common point is in general a quadric through the plane section common to S, U, V, W .

Solution by H. W. CURJEL, M.A.

Let $U \equiv S + AX, V \equiv S + AY, W \equiv S + AZ$; A, X, Y, Z being linear.

The six quadrics are

$$(SA_1)/(S_1A) = X/X_1 \text{ or } Y/Y_1 \text{ or } Z/Z_1 \text{ or } (X-Y)/(X_1-Y_1) \\ \text{or } (Y-Z)/(Y_1-Z_1) \text{ or } (Z-X)/(Z_1-X_1),$$

where the suffix denotes the substitution of the values of the coordinates of P .

These all pass through the two points given by

$$S/S_1 = (AX)/(A_1X_1) \text{ and } X/X_1 = Y/Y_1 = Z/Z_1,$$

i.e., through P and another point (Q say).

Let $S \equiv u + l + c$ and $A \equiv m + d$, and given plane locus of P be $n = 1$; u being a homogeneous function of the second degree in X, Y, Z , and l, m, n homogeneous functions of the first degree in X, Y, Z , and d and c being constants.

To find the coordinates of P in terms of those of Q , putting $X_1/X = x$,

we get
$$\frac{x^2u + xl + c}{u + l + c} = \frac{x^2m + xd}{m + d};$$

$$\therefore x = 1, \text{ giving point } Q; \text{ or } x = \frac{c(m+d)}{ud - ml - cm}, \text{ giving point } P;$$

$$\therefore 1/n = n_1/n = X_1/X = Y_1/Y = Z_1/Z = x = \frac{c(m+d)}{ud - ml - cm}.$$

Therefore locus of Q is

$$ud + ld + cd = ml + ld + cm + cd + (m+d)cn = (m+d)(l+c+nc),$$

or
$$Sd = A(l+c+nc),$$

a quadric through the common section (A, S) of S, U, V, W .

13523. (Professor SANJANA, M.A.)—Prove that

$$\frac{c_0}{m+n} - \frac{c_1}{m+3n} + \frac{c_2}{m+5n} - \dots + \frac{(-1)^l c_l}{m+(2l+1)n} \\ = \frac{2 \cdot 4 \cdot 6 \dots (2l-2) \cdot 2l \cdot n^l}{(m+n)(m+3n)(m+5n) \dots \{m+(2l+1)n\}},$$

where l, m, n are positive integers, and $c_0, c_1, c_2, \dots, c_l$ the coefficients in the expansion of $(1+x)^l$. Also deduce the identity

$$\frac{5}{9} - 4 \cdot \frac{5}{19} + 6 \cdot \frac{5}{29} - 4 \cdot \frac{5}{39} + \frac{5}{49} = \frac{5 \cdot 10 \cdot 20 \cdot 30 \cdot 40}{9 \cdot 19 \cdot 29 \cdot 39 \cdot 49}.$$

Solution by Rev. J. CULLEN, B.A.

$$\begin{aligned}\text{Let } S &= c_0 x^{m+n-1} - c_1 x^{m+2n-1} + \dots (-1)^l c_l x^{m+(2l+1)n-1} \\ &= x^{m+n-1} [c_0 - c_1 x^{2n} + \dots (-1)^l c_l x^{2ln}] \\ &= x^{m+n-1} (1 - x^{2n})^l.\end{aligned}$$

$$\begin{aligned}\text{Then } \Sigma &= \int_0^1 S dx = \int_0^1 x^{m+n-1} (1 - x^{2n})^l dx \\ &= \frac{1}{2n} \int_0^1 u^{(m-n)/(2n)} (1-u)^l du \quad (u = x^{2n}) \\ &= \frac{\Gamma(l+1) \Gamma[(m+n)/(2n)]}{2n \Gamma[l+1 + (m+n)/(2n)]} \\ &= \text{stated result on expanding and cancelling } \Gamma[(m+n)/(2n)].\end{aligned}$$

Putting $m = l = 4$ and $n = 5$, we have the identity on multiplying across by 5.

11954. (R. W. D. CHRISTIE.)—In triangular numbers, prove that, if $\Sigma = \frac{1}{2}(a+b+c+d)$, then

$$\begin{aligned}(1) \quad T_{a+n} + T_{b+n} + T_{c+n} + T_{d+n} &\equiv T_{\Sigma-d+n} + T_{\Sigma-c+n} + T_{\Sigma-b+n} + T_{\Sigma-a+n}; \\ \text{and } (2) \quad T_{a+n} + T_{b+n} + T_{c-n-1} + T_{d+n} \\ &\equiv T_{\Sigma+n} + T_{\Sigma+n-(c+d)} + T_{\Sigma+n-(b+c)} + T_{\Sigma+n-(a+c)},\end{aligned}$$

where a, b, c, d, n are any integers, and Σ not necessarily integral, and $T_x = \frac{1}{2}x(x+1)$.

Solution by R. KNOWLES, B.A.; Professor RADHAKRISHNAN; and others.

(1) By multiplication,

$$\begin{aligned}&\{(a+n)(a+n+1) + (b+n)(b+n+1) + (c+n)(c+n+1) + (d+n)(d+n+1)\}/2 \\ &= \{a^2 + b^2 + c^2 + d^2 + (2n+1)(a+b+c+d) + 4n(n+1)\}/2 \\ &= \{(\Sigma-d+n)(\Sigma-d+n+1) + (\Sigma-c+n)(\Sigma-c+n+1) \\ &\quad + (\Sigma-b+n)(\Sigma-b+n+1) + (\Sigma-a+n)(\Sigma-a+n+1)\}/2.\end{aligned}$$

(2)

$$\begin{aligned}&\{(a+n)(a+n+1) + (b+n)(b+n+1) + (c-n-1)(c-n) + (d+n)(d+n+1)\}/2 \\ &= \{a^2 + b^2 + c^2 + d^2 + (2n+1)(a+b+d-c) + 4n(n+1)\}/2 \\ &= \{(\Sigma+n)(\Sigma+n+1) + (\Sigma-c-d+n)(\Sigma-c-d+n+1) \\ &\quad + (\Sigma-b-c+n)(\Sigma-b-c+n+1) + (\Sigma-a-c+n)(\Sigma-a-c+n+1)\}/2.\end{aligned}$$

11907. (Professor MOREL.)—Le centre C d'un cercle se meut sur une parabole donne, et le cercle passe constamment par le foyer F de cette parabole. Trouver (1) l'enveloppe de ce cercle; (2) le lieu du point de

rencontre de ce cercle (*a*) avec une parallèle à l'axe menée par C, (*b*) avec l'ordonnée de C, (*c*) avec la tangente à la parabole en C, (*d*) avec la normale menée en C.

Solution by R. KNOWLES, B.A. ; Professor CHAKRIVARTI ; and others.

Let x_1, y_1 be the coordinates of C : the equation to the circle is

$$(x-x_1)^2 + (y-y_1)^2 = (a+x_1)^2.$$

Putting $x_1 = y_1^2/4a$, this becomes

$$2ax^2 - y_1^2x + 2ay^2 + ay_1^2 - 4ay_1y - 2a^3 = 0 \dots\dots\dots (1),$$

and its envelope breaks into the line $x+a=0$, the directrix, and a point

$$x^2 + y^2 - 2ax + a^2 = 0.$$

(*a*) In (1) put $y_1 = y$, and the required locus is the curve

$$2ax^2 - ay^2 - y^2x - 2a^3 = 0.$$

(*b*) In (1) put $x_1 = x$, and the locus is

$$x^2 - y^2 - 2ax + 4a^{\frac{1}{2}}x^{\frac{1}{2}}y + a^2 = 0.$$

(*c*) From (1) we get

$$y_1 = 2ay \pm [2a(a+x)\{y^2 + (a-x)^2\}]^{\frac{1}{2}}/(a-x),$$

and, substituting this in the equation to the tangent $(y_1-y)^2 = y^2 - 4ax$, the locus required is the curve

$$(a+x)[y + \{2a(x^2 + y^2 - 2ax + a^2)\}^{\frac{1}{2}}]^2 = (y^2 - 4ax)(a-x)^2.$$

(*d*) The equation to the normal is

$$y_1^3 - 4a(x-2a)y_1 - 8a^2y = 0.$$

Substituting the value of y_1 obtained above, we get the locus, which is complicated.

13526. (Professor W. BOOTH.)—If $\alpha, \beta, \gamma, \delta$ be the roots of the quartic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, and if all the expressions of the form $(2\alpha - \beta - \gamma)$ be taken and multiplied together, calculate the value of this product in terms of H, I, J, and G, the well-known functions of the differences of the roots.

Solution by W. E. JEFFARES, B.A.

By BURNSIDE and PANTON's *Theory of Equations* (first edition), pages 122-124 :

$$\left. \begin{aligned} M_1^2 &= a^2\theta_1 + b^2 - ac = a^2\theta_1 - H, \text{ since } H = ea - b^2, \\ M_2^2 &= a^2\theta_2 + b^2 - ac = a^2\theta_2 - H \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ M_3^2 &= a^2\theta_3 + b^2 - ac = a^2\theta_3 - H \quad \text{,,} \quad \text{,,} \quad \text{,,} \end{aligned} \right\} \dots\dots(1),$$

where $\theta_1, \theta_2, \theta_3$ are the roots of the reducing cubic
 $4a^2\theta^3 - Ia\theta + J = 0$

$$\left. \begin{aligned} a\alpha + b &= -M_1 + M_2 + M_3, & a\beta + b &= M_1 - M_2 + M_3, \\ a\gamma + b &= M_1 + M_2 - M_3, & a\delta + b &= -M_1 - M_2 - M_3 \end{aligned} \right\} \dots\dots(2),$$

where $\alpha, \beta, \gamma, \delta$ are the roots of

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

The twelve functions of the form $2\alpha - \beta - \gamma$ are thus easily expressed in terms of M_1, M_2, M_3 :—

$$\begin{aligned} \frac{1}{2}a(2\alpha - \beta - \gamma) &= M_2 + M_3 - 2M_1, & \frac{1}{2}a(2\beta - \gamma - \alpha) &= M_3 + M_1 - 2M_2, \\ \frac{1}{2}a(2\gamma - \alpha - \beta) &= M_1 + M_2 - 2M_3, & \frac{1}{2}a(2\alpha - \beta - \delta) &= M_3 + 2M_2 - M_1, \\ \frac{1}{2}a(2\beta - \alpha - \delta) &= M_3 - M_2 + 2M_1, & \frac{1}{2}a(2\delta - \alpha - \beta) &= -(M_1 + M_2 + 2M_3), \\ \frac{1}{2}a(2\alpha - \gamma - \delta) &= 2M_3 + M_2 - M_1, & \frac{1}{2}a(2\gamma - \alpha - \delta) &= M_2 - M_3 + 2M_1, \\ \frac{1}{2}a(2\delta - \gamma - \alpha) &= -(M_1 + 2M_2 + M_3), & \frac{1}{2}a(2\beta - \gamma - \delta) &= 2M_3 - M_2 + M_1, \\ \frac{1}{2}a(2\gamma - \beta - \delta) &= 2M_2 - M_3 + M_1, & \frac{1}{2}a(2\delta - \beta - \gamma) &= -(2M_1 + M_2 + M_3); \end{aligned}$$

These twelve can be reduced, by multiplying every two of them, to six, viz. :

$$\left. \begin{aligned} \frac{1}{4}a^2 \{2\gamma - \alpha - \beta\}(2\delta - \alpha - \beta) &= -\{(M_1 + M_2)^2 - 4M_3^2\} \\ &= -(M_1^2 + M_2^2 - 4M_3^2 + 2M_1M_2) = 5a^2\theta_3 - 2H - 2M_1M_2, \end{aligned} \right\} \text{from equations (1); therefore}$$

$$\left. \begin{aligned} \frac{1}{4}a^2(2\gamma - \alpha - \beta)(2\delta - \alpha - \beta) &= 5a^2\theta_3 - 2H - 2M_1M_2; \\ \text{similarly } \frac{1}{4}a^2(2\beta - \gamma - \alpha)(2\delta - \gamma - \alpha) &= 5a^2\theta_2 - 2H - 2M_3M_1, \\ \text{and } \frac{1}{4}a^2(2\alpha - \beta - \gamma)(2\delta - \beta - \gamma) &= 5a^2\theta_1 - 2H - 2M_2M_3, \\ \frac{1}{4}a^2(2\alpha - \gamma - \delta)(2\beta - \gamma - \delta) &= 5a^2\theta_3 - 2H + 2M_1M_2, \\ \frac{1}{4}a^2(2\alpha - \beta - \delta)(2\gamma - \beta - \delta) &= 5a^2\theta_2 - 2H + 2M_3M_1, \\ \frac{1}{4}a^2(2\beta - \alpha - \delta)(2\gamma - \alpha - \delta) &= 5a^2\theta_1 - 2H + 2M_2M_3 \end{aligned} \right\} \dots\dots\dots(3).$$

Again, by grouping the last six equations by two's, one out of the first three and one out of the last three, and multiplying them together, we get

$$\begin{aligned} \frac{1}{18}a^4(2\alpha - \beta - \gamma)(2\delta - \beta - \gamma)(2\beta - \alpha - \delta)(2\gamma - \alpha - \delta) &= (5a^2\theta_1 - 2H)^2 - 4M_2^2M_3^2 = (5a^2\theta_1 - 2H)^2 - 4(a^2\theta_2 - H)(a^2\theta_3 - H) \\ &= 25a^4\theta_1^2 - 24a^2H\theta_1 - 4a^4\theta_2\theta_3 = 25a^4\theta_1^2 - 24a^2H\theta_1 + aJ/\theta_1 \\ &= a/\theta_1 \{25a^3\theta_1^3 - 24aH\theta_1^2 + J\}; \\ \therefore \frac{1}{18}a^4(2\alpha - \beta - \gamma)(2\delta - \beta - \gamma)(2\beta - \alpha - \delta)(2\gamma - \alpha - \delta) &= a/\theta_1 \{25a^3\theta_1^3 - 24aH\theta_1^2 + J\}, \\ \frac{1}{18}a^4(2\beta - \gamma - \alpha)(2\delta - \gamma - \alpha)(2\alpha - \beta - \delta)(2\gamma - \beta - \delta) &= a/\theta_2 \{25a^3\theta_2^3 - 24aH\theta_2^2 + J\}, \\ \frac{1}{18}a^4(2\gamma - \alpha - \beta)(2\delta - \alpha - \beta)(2\alpha - \gamma - \delta)(2\beta - \gamma - \delta) &= a/\theta_3 \{25a^3\theta_3^3 - 24aH\theta_3^2 + J\} \end{aligned} \dots\dots(4).$$

Call now Π the product of the twelve functions of the form $2\alpha - \beta - \gamma$ and multiply the three equations of (4); we have

$$\begin{aligned} \frac{a^{12}}{4096} \Pi &= \frac{a^3}{\theta_1\theta_2\theta_3} [J^3 + \Sigma (25a^3\theta_1^3 - 24aH\theta_1^2) J^2 \\ &\quad + \Sigma (25a^3\theta_2^3 - 24aH\theta_2^2)(25a^3\theta_3^3 - 24aH\theta_3^2) J \\ &\quad + (25a^3\theta_1^3 - 24aH\theta_1^2)(25a^3\theta_2^3 - 24aH\theta_2^2)(25a^3\theta_3^3 - 24aH\theta_3^2)]; \end{aligned}$$

$$\therefore \frac{a^6}{4096} \Pi = -\frac{4}{J} [J^3 + \Sigma (25a^2\theta_1^3 - 24aH\theta_1^2)J^2 + \Sigma (25a^2\theta_2^3 - 24aH\theta_2^2)(25a^2\theta_3^3 - 24aH\theta_3^2)J + (25a^2\theta_1^3 - 24aH\theta_1^2)(25a^2\theta_2^3 - 24aH\theta_2^2)(25a^2\theta_3^3 - 24aH\theta_3^2)] \dots (5).$$

$$\begin{aligned} \text{Now } \theta_1 + \theta_2 + \theta_3 &= 0, \quad \Sigma (\theta_1\theta_2) = -\frac{I}{4a^2}, \quad \theta_1\theta_2\theta_3 = -\frac{J}{4a^3}, \\ \Sigma (\theta_1^2) &= \frac{I}{2a^2}, \quad \Sigma (\theta_1^3) = -\frac{3J}{4a^3}, \quad \Sigma (\theta_1^2\theta_2^2) = \frac{I^2}{16a^4}, \\ \Sigma (\theta_1^3\theta_2^2) &= -\frac{IJ}{16a^5}, \quad \Sigma (\theta_1^3\theta_2^2\theta_3^2) = -\frac{IJ^2}{64a^5}, \quad \Sigma (\theta_1^3\theta_2^2\theta_3^2) = 0, \quad \Sigma (\theta_1^3\theta_2^2) = \frac{12J^2 - I^3}{64a^6}, \\ \Sigma (25a^2\theta_1^3 - 24aH\theta_1^2) &= 25a^2\Sigma\theta_1^3 - 24aH\Sigma\theta_1^2 = -\frac{75aJ + 48HI}{4a} \\ &= -\frac{1200a^3J + 768a^2HI}{64a^3}, \\ \Sigma (25a^2\theta_1^3 - 24aH\theta_1^2)(25a^2\theta_2^3 - 24aH\theta_2^2) &= 625a^4\Sigma (\theta_1^3\theta_2^3) - 600a^4H\Sigma (\theta_1^3\theta_2^2) + 576a^2H^2\Sigma (\theta_1^2\theta_2^2) \\ &= \frac{7500a^3J^2 - 625a^2I^3 + 2400a^2HIJ + 2304aH^2I^2}{64a^3}, \\ (25a^2\theta_1^3 - 24aH\theta_1^2)(25a^2\theta_2^3 - 24aH\theta_2^2)(25a^2\theta_3^3 - 24aH\theta_3^2) &= 15625a^2\theta_1^3\theta_2^3\theta_3^3 - 15000a^2H\Sigma (\theta_1^3\theta_2^3\theta_3^2) \\ &\quad + 14400a^2H^2\Sigma (\theta_1^3\theta_2^2\theta_3^2) - 13824a^2H^3\theta_1^3\theta_2^2\theta_3^2 \\ &= \frac{15000a^2HIJ^2 - 15625a^3J^2 - 55296H^3J^2}{64a^3}. \end{aligned}$$

From (5), by putting in for the several symmetric functions their terms in H, I, J, we have

$$\begin{aligned} \frac{a^6\Pi}{4096} &= -\frac{4}{J} \left[J^3 - \frac{1200a^3J^3 + 768a^2HIJ^2}{64a^3} \right. \\ &\quad + \frac{7500a^3J^2 - 625a^2I^3 + 2400a^2HIJ^2 + 2304aH^2I^2}{64a^3} \\ &\quad \left. + \frac{15000a^2HIJ^2 - 15625a^3J^2 - 55296H^3J^2}{64a^3} \right], \end{aligned}$$

which, when simplified, becomes

$$a^2\Pi = 256 \{ 9261a^3J^2 - 16632a^2HIJ + 625a^2I^3 - 2304a^2H^2I^2 + 55296H^3J \}.$$

13585. (Rev. T. MITCHESON, B.A.)—If a, b, c be the sides of a triangle, x, y, z the medians, and l, m, n the perpendiculars from the respective angular points upon the sides, then

$$4(x^2 + y^2 + z^2) = 3(l^2b^2m^2 + m^2c^2n^2 + n^2a^2l^2).$$

Solution by E. WALTER TODD, JOHN SADLER, and many others.

$$\begin{aligned}
 \text{Dexter side} &= 3 \{ (4\Delta^2/l^2) + (4\Delta^2/m^2) + (4\Delta^2/n^2) \}, \\
 \text{when } \Delta &= \text{area of } \Delta, \text{ for } al = bm = cn = 2\Delta \\
 &= 3 \{ (2\Delta/l)^2 + (2\Delta/m)^2 + (2\Delta/n)^2 \} \\
 &= 3 \{ a^2 + b^2 + c^2 \}, \text{ for } al = \Delta \text{ as before} \\
 &= 4 (x^2 + y^2 + z^2), \text{ by well-known theorem} \\
 &= \text{sinister side.}
 \end{aligned}$$

[Mr. ANGUS MACMURCHY solves the Question as follows:—

$$\begin{aligned}
 x^2 &= \frac{1}{4} (c^2 + a^2) - \frac{1}{2} (2ac) \cos B = \frac{1}{4} (2c^2 - a^2 + 2b^2), \text{ \&c.}, \\
 \text{and } l^{-1}m &= c/v, \text{ \&c. Substitute in proposed identity, which then becomes} \\
 4 \cdot \frac{3}{4} (a^2 + b^2 + c^2) &= 3 (a^2 + b^2 + c^2).]
 \end{aligned}$$

12646. (Professor ESCOTT.)—Through a point O on the produced diameter AB of a semicircle draw a secant ORR', so that the quadrilateral ABR'R' inscribed in the semicircle shall be a maximum. Prove that, in this case, the projection of RR' on AB is equal in length to the radius of the circle.

Solution by Professor COLAW.

Denote the centre of the circle by C, the angle ROC by θ , the radius BC by R, the distance OC by d . Then

$$OR' = d \cos \theta + (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}}, \quad OR = d \cos \theta - (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}},$$

$$OA = d + R, \quad \text{and} \quad OB = d - R.$$

Also, area of quadrilateral ABR'R' is $\frac{1}{2} (OA \cdot OR' - OB \cdot OR) \sin \theta$, or, substituting and reducing, $d [R \cos \theta + (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}}] \sin \theta$. Equating the first derivative with regard to θ to zero and reducing,

$$(R^2 - d^2 \sin^2 \theta) \cos^2 \theta + \frac{1}{2} [R (1 - 2 \sin^2 \theta)] (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta = \frac{1}{2} R \cos^2 \theta.$$

Regarding $(R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta$ as the unknown quantity, this equation is in the quadratic form. Solving,

$$(R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta = \frac{1}{2} R, \quad \text{or} \quad 2 (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta = R.$$

$$\text{But} \quad RR' (= OR' - OR) = 2 (R^2 - d^2 \sin^2 \theta)^{\frac{1}{2}}.$$

Therefore the first member of the last equation is the projection of RR' on AB, and is equal to the radius of the circle. Solving for θ ,

$$\cos 2\theta = [-R^2 \pm (d^4 - d^2 R^2 + R^4)^{\frac{1}{2}}]/d^2.$$

It may be shown that the negative sign before the radical must be rejected.

12675. (Professor DOUGNAC.)—Si $a^3 + b^3 + c^3 = 0$, démontrer que
 $a + b + c = 3abc / \{a(b-a) + b(c-b) + c(a-c)\}.$

Solution by Rev. S. J. ROWTON, M.A., Mus.D.; R. CHARTRES; and others.

$$a^3 + b^3 + c^3 - 3abc = (a + b + c) \{a(a-b) + b(b-c) + c(c-a)\}.$$

Therefore, if

$$a^3 + b^3 + c^3 = 0, \quad a + b + c = 3abc / \{a(b-a) + b(c-b) + c(a-b)\}.$$

13576. (Professor M. W. CROFTON, F.R.S.)—If $4n-1$ is a prime number, then $1 \cdot 2 \cdot 3 \dots (2n-1) \pm 1$ is divisible by $4n-1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Show that $\frac{p-1}{2}! \pm 1$ is divisible by p , when $p = 4n-1 = \text{prime}$.

WILSON'S theorem gives $(p-1)! \equiv -1, (\text{mod } p)$. Now $(p-1)!$ may be written as the product of the two *half* factorials of $\frac{1}{2}(p-1)$ terms each

$$[1 \cdot 2 \cdot 3 \dots \frac{1}{2}\{p-1\}][\{p-1\}(p-2)(p-3) \dots \{p-\frac{1}{2}\}(p-1)],$$

and the terms of the latter product are congruent to

$$-1, -2, -3, \dots -\frac{1}{2}(p-1), (\text{mod } p);$$

$$\text{therefore} \quad (p-1)! \equiv \frac{p-1}{2}! \left[(-1)^{\frac{1}{2}(p-1)} \frac{p-1}{2}! \right], (\text{mod } p)$$

$$\equiv (-1)^{\frac{1}{2}(p-1)} \left(\frac{p-1}{2}! \right)^2$$

$$\equiv - \left(\frac{p-1}{2}! \right)^2, (\text{mod } p), \text{ since } p = (4n-1);$$

$$\text{therefore} \quad \left(\frac{p-1}{2}! \right)^2 \equiv +1, \text{ and } \frac{p-1}{2}! \equiv \pm 1, (\text{mod } p),$$

which shows that $\frac{p-1}{2}! \mp 1$ is divisible by p .

13628. (J. C. WASSON.)—When a variable point moves round the circumcircle of any triangle, the motion of the SIMSON'S line with respect to that point is identical with that of a chord of a circle one extremity of which passes over twice the arc described by the other, the arcs being measured in opposite directions.

intersecting each other at the orthocentre O. X, Y, and Z are the mid-points of the sides, and let the nine-point circle be described round X, Y, Z, P, Q, and R.

Let NLM be the SIMSON's line with respect to any point K on the circumcircle of ABC, and let L'M'N' be the SIMSON's line with respect to K', the point on the circumcircle diametrically opposite to K.

By a well-known theorem, NLM and L'M'N' intersect at right angles on the nine-point circle. Call the point of concurrence T.

Now, since L, L' are the feet of perpendiculars on BC from two diametrically opposite points on the circumcircle, L and L' are equidistant from the mid-point of BC. Therefore $LX = XL'$. But $\angle LTL'$ is a right angle. Therefore $LX = XT$. Therefore, by lemma,

$$\text{arc PT} = 2 \text{ arc XD}.$$

Now, let H be the point in the arc of the nine-point circle such that $\text{arc XH} = \frac{1}{2} \text{ arc XP}$. H is thus a fixed point.

Now $\text{arc PT} = 2 \text{ arc XD}$ and $\text{arc HP} = 2 \text{ arc HX}$, by construction ; therefore $\text{arc HT} = 2 \text{ arc HD}$.

Therefore DT is a chord of the nine-point circle, the extremity T passing over double the arc described by D, as the line moves about.

Corollary 1.—Let I, J be points in the arcs YQ, RZ such that

$$\text{arc QI} = 2 \text{ arc IY} \quad \text{and} \quad \text{arc RJ} = 2 \text{ arc JZ}.$$

Now angle subtended at centre by arc ZX = $2 \angle XYZ = 2 \angle B$. Also angle subtended at centre by arc XP

$$\begin{aligned} &= 2 \angle XYP = 2 \angle XYC - 2 \angle PYC = 2A - 2(\pi - 2C) \\ &= 2(\pi - B - C) - 2(\pi - 2C) = 2(C - B). \end{aligned}$$

Similarly, angle subtended at centre by arc ZR = $2(A - B)$; therefore angle subtended at centre by arc JDH

$$= 2B + \frac{2}{3}(A - B) + \frac{2}{3}(C - B) = \frac{2}{3}(A + B + C) = \frac{2}{3}\pi.$$

Similarly,

$$\text{arc HI} = \text{arc IJ} = \text{arc JH}.$$

Corollary 2.—H, I, and J are the points at which the SIMSON's line becomes tangential to the nine-point circle. The points D, T will coincide if HD is zero, or if $\text{arc HD} + \text{arc HT}$ is a multiple of 2π . In the former case the SIMSON's line touches the nine-point circle at H. In the latter case arc HD is a multiple of $\frac{2}{3}\pi$, and the line becomes tangential at I or J.

[Mr. DORRIS gives the following short proof of this "very pretty" proposition:—"Lemma.—If any SIMSON-line of a triangle ABC meets the base BC in L and the altitude AP in α , then $L\alpha$ is bisected by the N.P. circle. Let O be the orthocentre, K the point on the circumcircle corresponding to the SIMSON-line. Then it is well known that KLO α is a parallelogram, the diagonals of which bisect one another at D. Also D, being the mid-point of OK, is a point on the N.P. circle. Now let a SIMSON-line meet the base BC in L, the altitude AP in α , and the N.P. circle at the mid-point D of $L\alpha$ and also at T. Then $\angle PDT = 2 \angle LPD$, Therefore $\text{arc PT} = 2 \text{ arc XD}$, where X is the mid-point of the base. Hence the proposition follows.]

13623. (R. CHARTRES.)—Two random points P, Q are taken within the triangle ABC; find the mean value of the triangle APQ.

Solution by the PROPOSER.

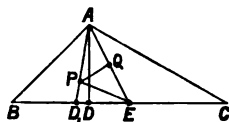
Let D, E be two random points in BC, and D, indefinitely near to D; then the mean value of ADE = $\frac{1}{3}\Delta$. Hence, if P moves in the element AD, the mean value of

$$APE = \frac{2}{3} \text{ of } ADE = \frac{2}{9}\Delta,$$

since the mean value of AP = $\frac{2}{3}AD$. And, if Q moves similarly, the mean value of

$$APQ = \frac{2}{3} \text{ of } APE = \frac{4}{27}\Delta.$$

[Otherwise.—Divide BC into equal infinitesimal parts, by points which, joined to A, divide ABC into equal infinitesimal triangles. P and Q likewise will fall on any of these triangles with equal probability, and the average distance from A will be two-thirds of the distance to the infinitesimal base. The average distance apart of the bases is $\frac{1}{3}BC$. Therefore the average area of APQ = $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$ of the given triangle.]



13618. (ROBERT W. D. CHRISTIE.)—Where P is any odd prime whatsoever and Q an integral function of that prime, prove

$$X^2 - 5Y^2 = PQ + 1,$$

and find general integral values for X and Y.

Solution by H. W. CURJEL, M.A.

It is easily seen that

$$\left\{\frac{1}{2}(5p \pm 3)\right\}^2 - 5 \left\{\frac{1}{2}(p \pm 1)\right\}^2 = p(5p \pm 5) + 1.$$

[The PROPOSER adds the following :—Another simple value is

$$\left\{\frac{3}{2}(p \pm 1)\right\}^2 - 5 \left\{\frac{1}{2}(p \pm 1)\right\}^2 = p(p + 2) + 1,$$

and a general value is

$$X = \frac{1}{2}(ap \pm 3)^2, \quad Y = (bp \pm 1)^2, \quad q = p \cdot (a^2 - 5b^2)/4 \pm \{2(3a - 5b)\}/4,$$

where a and b are arbitrary so long as they are of the same affection. But my special object in the above question was to draw attention to a remarkable property of the "continuant" series:—1, 1, 2, 3, 5, &c.

Let a, b be any two consecutive terms of this series, then

$$(a + 2b)^2 - 5b^2 = p \left\{ 1 + \frac{p-3}{2!} + \frac{p-4 \cdot p-5}{3!} + \&c. \right\} + 1$$

when and when only p is prime.

Ex. gr.—Let a = 3, b = 2, then

$$7^2 - 5 \cdot 2^2 = 7 \{1 + 2 + 1\} + 1.$$

Here 3 is the 4th term of the series, and $p = 7 = 2 \times 4 - 1$.

Again, let $a = 5$, $b = 3$; then $11^2 - 5 \cdot 3^2 = 76$. Here $a = 5$; and, since $2 \times 5 - 1 = 9$, which is *not* a prime, the equation fails.—*Vide Nature*, May 6, 1897.]

13627. (W. J. DOBBS, M.A.)—The straight line AB is produced to P, so that rect. AP . PB = sq. on AB. In the straight line which bisects AB in C at right angles, a point Q is taken such that BQ = AP. Prove that QB bisects the angle CQP.

Solutions (1) by Rev. W. J. CONSTABLE, M.A.; (2) by Rev. S. J. ROWTON, M.A., Mus.D.

(1) Construct as required. With centre A, describe circle PDX; place PD in it equal to AB. Describe circle ABD.

Then, as in iv. 10, APD is a triangle having each of its angles at base double of vertical angle, and it may easily be shown that Q lies at intersection of the two circles [BQ = AP and BQ = AQ, &c.]

Hence, it follows

$$\angle PQD = \frac{1}{2}PAD, \quad \angle BQD = \angle BAD;$$

$$\therefore \angle BQP = \angle \frac{1}{2}PAD = \angle CQB,$$

for $\triangle QAB$ and $\triangle APD$ are equal in all respects.

$$(2) CP^2 = AP \cdot PB + CB^2$$

(Euc. II. 10)

$$= AB^2 + CB^2$$

$$= 5CB^2;$$

therefore, if we take CB as the unit, the other lines will be to one another as indicated in the figure; and hence it will be found that

$$CQ : QP :: CB : BP;$$

therefore BQ bisects the angle CQP (vi. 3).

[Mr. REAY solves the Question as follows:—

Join AQ. On BQ take K, such that BK = BA. Then

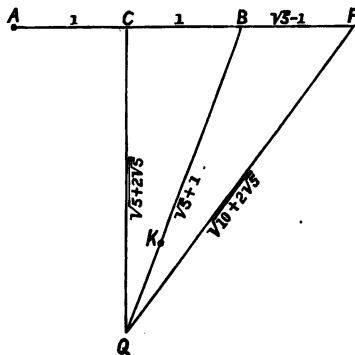
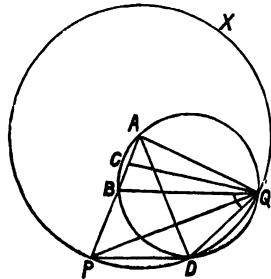
$$QK = BP;$$

therefore

$$QK \cdot QB = KB^2 = AB^2;$$

therefore, by Euclid iv. 10, $\angle ABQ = 2\angle AQB$. Put $\angle CQB = \theta$, $\angle BQP = \phi$;

$$\angle APQ = \angle AQP = 2\theta + \phi; \text{ also } \angle ABQ - \angle BQP = 4\theta - \phi;$$



therefore $2\theta + \phi = 4\theta - \phi$; therefore $\phi = \theta$;
therefore QB bisects CQP.

But "W. J. R.," an ingenious correspondent in West Kensington, gives the following improvement on all the foregoing:—"No further construction than what is directed in the enunciation is needful. For AQB is the isosceles triangle of rv. 10. Therefore QAB is two-fifths of two right angles, and AQB is two-tenths of the same. But $AQ = BQ = AP$. Therefore AQP is three-tenths of two right angles, and BQP is one-tenth of the same. Hence, &c.]

13629. (R. F. MUIRHEAD.)—Prove that the expression

$$\alpha(\alpha-1)(\alpha-2) \dots (\alpha-r+1) + r \cdot \alpha(\alpha-1) \dots (\alpha-r+2) \beta$$

$$+ \frac{r(r-1)}{1 \cdot 2} \alpha(\alpha-1) \dots (\alpha-r+3) \beta(\beta-1) + \dots$$

$$+ \beta(\beta-1)(\beta-2) \dots (\beta-r+1)$$
 is unaltered in value when we change α into $\alpha+1$ and β into $\beta-1$.

Solutions (1) by LIONEL E. REAY, B.A., and H. W. CUFJEL, M.A.;
 (2) by Prof. T. SAVAGE, M.A.; (3) by Rev. J. CULLEN, B.A.

(1) By VANDERMONDE's theorem, the given expression

$$= (\alpha + \beta)(\alpha + \beta - 1)(\alpha + \beta - 2) \dots (\alpha + \beta - r + 1).$$

This is a function of $\alpha + \beta$ only, and is therefore unchanged when α and β vary subject to the condition $\alpha + \beta = \text{const.}$ The question proposed is thus only a particular case of VANDERMONDE's theorem.

(2) For α write $\beta-1$, and for β write $\alpha+1$ in the given expression; then write the resulting series in the reverse order.

(3) Let the expression $= u_{\alpha, \beta}$. Then

$$\Delta_{\alpha} u_{\alpha, \beta} = \Delta_{\beta} u_{\alpha, \beta}; \text{ therefore } (1 + \Delta_{\alpha}) u_{\alpha, \beta} = (1 + \Delta_{\beta}) u_{\alpha, \beta},$$

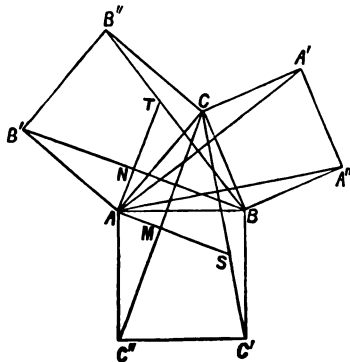
or

$$u_{\alpha, \beta} = E_{\alpha} E_{\beta}^{-1} u_{\alpha, \beta} = u_{\alpha+1, \beta-1}.$$

[Professor SANJANA puts the following query:—Does the proportion hold good when α is fractional, and when $\alpha-r+1$ is negative? For a similar result in combinations, where repetitions are allowed, see Quest. 13403.

The PROPOSER, in reply to Professor SANJANA's question, says: "The identity *does* hold for *all* values of α and β , though by the form of the question r is necessarily a positive integer. This follows at once from the fact that the identity depends on VANDERMONDE's Theorem." And he adds: "Professor SAVAGE's reply indicates a simple way of proving VANDERMONDE's Theorem by mathematical induction when the numbers concerned are integral."]

13481. (W.S. COONEY, B.A. T.C.D.) — If squares $AB'B'C$, $CA'A''B$, $BC'C''A$ be described on sides of any triangle ABC , and AA' , AA'' , BB' , BB'' , CC' , CC'' be drawn from each angle of triangle to opposite corners of squares, then (1) triangle formed by intersections of AA' , BB' , CC' equals triangle formed by AA'' , BB'' , CC'' ; (2) the three circles round quadrilaterals formed by AA' , AA'' , CC' , BB'' ; BB' , BB'' , AA' , CC'' ; CC' , CC'' , BB' , AA'' have a common point O , which connects collinearly with any angle of the triangle A , the centre of opposite square, and intersection of CC'' and BB' ; (3) if perpendiculars AM , AN be drawn to CC'' and BB' , and produced to meet CC' and BB'' in S and T , then $AS = AT$, and O is centre of completed square AST , and of each square similarly described at B and C ; (4) MN is SIMSON'S line for point A for lines BB' , CC'' , BC'' , CB' .



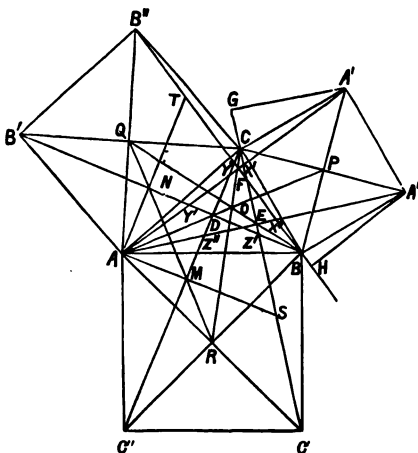
Solution by H. W. CURJEL, M.A.

Let P , Q , R be the centres of the squares on BC , CA , AB ; and CC' cut AA' , AA'' , BB' in X' , E , Z' ; CC'' cut BB'' , BB' , AA'' in Y'' , D , Z'' ; AA' cut BB' , BB'' in Y' and F ; and AA'' cut BB' in E .

$\triangle QAR$ is clearly similar to the two equal $\triangle s$ $B'AB$, CAC'' ; and

$AQ : AB' = 1 : \sqrt{2}$,
and each of the angles which the sides of $\triangle QAR$ make with the corresponding sides of $\triangle s$ $B'AB$, $CAC'' = \frac{1}{2}$ right angle; therefore QR is a diagonal of the square of which AD is the other diagonal (i.e., of $\triangle ANDM$), since $AN = AM$.

Therefore QR passes through MN ; therefore MN is SIMSON'S line of A for the lines BB' , CC'' , BC'' , CB' (4).



Draw $A'G$, $A''H$ perpendicular to CC' , $B''B$. Since $X'FEX''$ are concyclic,

$$\angle GX'A' = HX''A''.$$

Therefore $AX' : A''X'' = A'G : A''H = AA' : AA''$;

therefore $X'X''$ is parallel to $A'A''$.

Therefore, if AP cuts the circle $X'FEX''$ in O , $\angle X'OX''$ is a right angle, and $OX' = OX''$; therefore $\angle OEX'$ is $\frac{1}{2}$ right angle; therefore BEO is a straight line. But, since A, Q, C, E are concyclic, $\angle CEQ$ is $\frac{1}{2}$ right angle; therefore $BEOQ$ is a straight line. Similarly, $CFOR$ is a straight line. Therefore AP, BQ, CR pass through O (the orthocentre of $\triangle PQR$). Similarly, circles $Z'DEZ', Y'DFY''$ pass through O (2). And $\angle Z'OZ'' = \angle Y'OY'' =$ a right angle, and $OZ' = OZ''$ and $OY' = OY''$; therefore $\triangle X'Y'Z', X''Y''Z''$ are equal, for they can be made to coincide by turning one of them through a right angle about O ... (1).

Since $\triangle X'Y'Z' = \triangle X''Y''Z''$ and $AN = AM$,

therefore $\triangle X'SA = \triangle X'AT$; therefore $SA = AT$.

Again, $\angle ATF = \angle DY''F =$ supplement of $\angle AOF$; therefore A, O, F, T are concyclic, and $\angle TOA$ is a right angle.

Similarly, $\angle AOS$ is a right angle; therefore TOS is a straight line, and O is the centre of the completed square TAS .

Similarly, it may be shown that O is the centre of the squares similarly described at B and C .

13622. (R. F. DAVIS, M.A.)—Prove that the value of the quasi-continued fraction

$$\frac{12}{\sqrt{10+}} \frac{12}{\sqrt{10+}} \frac{12}{\sqrt{10+}} \dots \text{ad inf.}$$

is $12 \div 3.64575134 = 3.2915026$.

Solution by H. W. CURJEL, M.A.

Let value $= x$. Then $x = \frac{12}{\sqrt{10+x}}$;

therefore $x^3 + 10x^2 - 144 = 0$, or $(x+6)(x^2 + 4x - 24) = 0$.

The positive root of this

$$= \sqrt{28} - 2 = 5.2915026 - 2 = 3.2915026.$$

13616. (JOHN SADLER.)—Find the locus of the centres of all circles which touch two fixed (unequal) circles (which already do not touch one another).

Solution by LIONEL E. REAY, B.A. ; *and* W. E. JEFFARES, B.A.

(1) Circle touching circles O and O' externally,

$$PO - a = PO' - b;$$

$$\therefore PO - PO' = a - b;$$

therefore locus of P is one branch of the hyperbola whose foci are O and O', and transverse axis $a - b$.

(2) Touching both internally,

$$PO + a = PO' + b;$$

$$\therefore PO' - PO = a - b;$$

therefore locus of P is the other branch of this hyperbola.

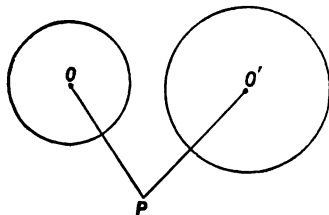
(3) Touching O internally,

$$PO + a = PO' - b; \text{ therefore } PO' - PO = a + b;$$

therefore locus is one branch of hyperbola, foci O and O', and transverse axis $a + b$.

(4) Touching O' internally, locus of P is the other branch of this hyperbola.

[Mr. CURJEL solves the Question as follows :—The locus is clearly an hyperbola with foci at the centres of the circles and transverse axis equal to the difference of the radii ; except in the case when one circle is entirely within the other, when the locus is an ellipse, major axis equal to the sum of the radii.]



7562. (R. KNOWLES, M.A.)—Prove, and verify (when $x = 0$), that

$$1 + \frac{1+x}{1.2} + \frac{(1+x)(1+2x)}{1.2.3} + \&c. \text{ ad inf. } \equiv (1+x)^{1/x} - 1.$$

Solution by H. J. WOODALL, A.R.C.S.

$$(1-x)^{-1/x} = 1 + (-1/x)(-x) + \&c.$$

$$= 1 + 1 + (1+x)/(1.2) + (1+x)(1+2x)/(1.2.3) + \&c.$$

When $x = 0$, this is the exponential theorem.

5154. (Professor TOWNSEND, F.R.S.)—A cone of revolution of uniform density being supposed to float with vertex downwards and axis vertical in a gravitating fluid of greater density ; show that the sphere having its

axis for diameter will intersect its surface in the limiting circle of flotation consistent with the stability of its equilibrium under the action of gravity.

Solution by Rev. J. CULLEN, B.A.

Let OP be the portion of the axis OQ immersed.

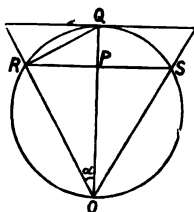
Then $Ak^3 = \pi OP^4 \tan^4 \alpha/4$,

$V = \pi OP^3 \tan^3 \alpha/3$; $\therefore HM = 3OP \tan^2 \alpha/4$,

and $HG = \frac{1}{2}OQ - \frac{1}{2}OP$.

Therefore equilibrium is stable or unstable according as $OP \tan^2 \alpha >$ or $< OQ - OP$; in other words, the distance OP of the limiting circle from the vertex = $OQ \cos^2 \alpha$. But, from the figure, $OP = OR \cos \alpha = OQ \cos^2 \alpha$.

Hence the question.



989. (M. COLLINS, B.A.)—If $a = \sqrt[n]{N}$ nearly, prove that $\sqrt[n]{N}$ will be $= \frac{(n+1)N + (n-1)a^n}{(n-1)N + (n+1)a^n} \times a$, still more nearly. Prove that each new application of this well-known formula will give a new root, having in general three times as many correct figures as the foregoing approximate root from which it is found.

Solution by Professor SANJANA, M.A.

Let $\sqrt[n]{N} = a + x$; then the ratio of the error committed in taking a for the root to a is x/a . If the expression to be proved = b , the error committed is $a + x - b = x - (b - a)$. Now

$$\begin{aligned} \frac{b-a}{a} &= \frac{(n+1)N + (n-1)a^n}{(n-1)N + (n+1)a^n} - 1 = \frac{2N - 2a^n}{(n-1)N + (n+1)a^n} \\ &= \frac{2(1+x/a)^n - 2}{(n-1)(1+x/a)^n + n + 1} \\ &= \frac{2n \frac{x}{a} + 2 \frac{n(n-1)}{1 \cdot 2} \frac{x^2}{a^2} + 2 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \frac{x^3}{a^3} + \dots}{2n + n(n-1) \frac{x}{a} + \frac{n(n-1)(n-1)}{1 \cdot 2} \frac{x^2}{a^2} + \dots} \\ &= \frac{\frac{x}{a} + \frac{n-1}{2} \frac{x^2}{a^2} + \frac{(n-1)(n-2)}{6} \frac{x^3}{a^3} + \dots}{1 + \frac{n-1}{2} \frac{x}{a} + \frac{(n-1)(n-1)}{4} \frac{x^2}{a^2} + \dots} \\ &= \frac{x}{a} - \frac{(n-1)(n+1)}{12} \frac{x^3}{a^3} + \dots, \end{aligned}$$

by long division or binomial theorem. Hence

$$\frac{a+x-b}{a} = \frac{(n-1)(n+1)}{12} \frac{x^3}{a^3} + \dots$$

Thus the chief term in the new error varies as the cube of the original one. If x/a is a decimal beginning with m ciphers, x^3/a^3 will in general begin with $3m$ ciphers. Hence the theorem.

13571. (Rev. W. ALLEN WHITWORTH, M.A.)—A stick 18 inches long is divided at random into three parts. John is to have the longest piece, Tom the next, and Sam the shortest. Determine their expectations.

Solutions (1) by Rev. T. ROACH, M.A.; (2) by Rev. T. C. SIMMONS, M.A.

(1) A line 18 inches long is divided into three by two chance cuts. Find the proportionate value of the three portions.

Let D be the first cut; then the chance of the size $\frac{A}{18}$ $\frac{D}{18}$ $\frac{B}{18}$ of the largest piece is $\frac{1}{3}$.

Let AD be the largest, and let E be the next cut. Then chance of E falling in AD is 3 : 1.

If E falls on AD, value of largest piece = $\frac{3}{4} \times \frac{1}{3}$,

“ “ DB, “ “ “ = $\frac{1}{4}$;

$$\therefore \text{value of largest piece} = \frac{3 \times \frac{1}{4} \times \frac{1}{3} + 1 \times \frac{1}{4}}{3 + 1} = \frac{1}{2}.$$

If E falls on AD, value of second piece = $\frac{1}{4}$,

“ “ DB, “ “ “ = $\frac{1}{4} \times \frac{1}{3}$;

$$\therefore \text{value of second piece} = \frac{3 \times \frac{1}{4} + 1 \times \frac{1}{4} \times \frac{1}{3}}{3 + 1} = \frac{1}{4};$$

$$\therefore \text{value of third piece} = \frac{1}{4};$$

therefore ratio 39 : 15 : 10.

(2) The following solution is offered with some hesitation, but I believe it to be correct.

Let x = length of John's portion, y = length for Sam. Then limits of x are 18 to 6, limits of y are $\frac{1}{3}(18-x)$ to 0. It will not affect the generality of the question if we suppose x to be always measured from one definite end of the line, and y always measured from the other end.

$$\text{The mean value of } x = \int_6^{18} \int_0^{\frac{1}{3}(18-x)} x \, dx \, dy \div \int_6^{18} \int_0^{\frac{1}{3}(18-x)} dx \, dy = 10.$$

$$\text{Mean value of } y = \int_6^{18} \int_0^{\frac{1}{3}(18-x)} y \, dx \, dy \div \int_6^{18} \int_0^{\frac{1}{3}(18-x)} dx \, dy = 2.$$

Hence John's expectation is 10 inches, Sam's 2 inches, and consequently Tom's expectation (as is also evident beforehand) 6 inches.

[Mr. WHITWORTH, having seen the above solutions, makes the following remarks:—Mr. ROACH's method seems to me to be altogether unsound. He appears to assume that $E(x) \times E(y) = E(xy)$, which implies that $\Sigma(x) \times \Sigma(y) = \Sigma(xy)$,

and this is altogether wrong.

Mr. SIMMONS's solution also appears to be wrong. He says that, the limits of x being 18 to 6, the limits of y are $\frac{1}{2}(18-x)$ to 0. But this is only true when $x > 12$. Suppose $x = 7$ and $y = 1$; Mr. SIMMONS's limits are satisfied, but the conditions of the problem are not satisfied, for Tom would have more than John.

Our results may be compared as follows—

ROACH	11.0		4.2		2.8
SIMMONS	10.0		6.0		2.0
WHITWORTH.....	11.0		5.0		2.0

My solution is No. 680 in my volume of *DCC Exercises in Choice and Chance*, just published by Deighton, Bell, & Co.]

13602. (Rev. J. CULLEN, B.A.)—Given four coplanar closed polygons whose sides are $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_r, d_1, d_2, \dots, d_s$, show that

$$(\Phi_{ab}^2 + \Psi_{ab}^2)(\Phi_{cd}^2 + \Psi_{cd}^2) = \Theta_a \Theta_b \Theta_c \Theta_d,$$

where

$$\Phi_{ab} = \Sigma a_m b_n \cos \phi_{m,n}, \quad \Psi_{ab} = \Sigma a_m b_n \sin \phi_{m,n},$$

$$\Theta_a = \Sigma a_m^2 + 2 \Sigma a_m a_n \cos \theta_{m,n},$$

excluding in each case a_p, b_q, c_r , and d_s , $\phi_{m,n}$ and $\theta_{m,n}$ being the angles between a_m and b_n , a_m and a_n .

Solution by the PROPOSER.

Let $a_1, a_2, \dots, a_p, \beta_1, \beta_2, \dots, \beta_q, \gamma_1, \gamma_2, \dots, \gamma_r$, and $\delta_1, \delta_2, \dots, \delta_s$, be the vector sides of the polygons. Then, since the polygons are closed,

$$a_p = a_1 + a_2 + \dots + a_{p-1};$$

similarly for β_q, γ_r , and δ_s . Now, by a known transformation in quaternions,

$$S^2 a_p \beta_q + T^2 V a_p \beta_q = T^2 a_p T^2 \beta_q,$$

$$S^2 \gamma_r \delta_s + T^2 V \gamma_r \delta_s = T^2 \gamma_r T^2 \delta_s.$$

Whence, multiplying, we have the result in the question, since

$$S^2 a_p \beta_q = \Phi^2 ab, \quad T^2 V a_p \beta_q = \Psi^2 ab, \quad T^2 a_p = \Theta_a.$$

5102. (Rev. G. H. HOPKINS, M.A.)—From any point on the surface of a circular cylinder planes are drawn. Find the equation to the surface upon which the foci of all the elliptic sections are placed; and prove

that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the Logocyclic Curve.

Solution by Rev. J. CULLEN, B.A.

Since the point of contact of the inscribed sphere with the plane determines a focus, we have $(x-a)^2 + y^2 + (z-\gamma)^2 = a^2$ (1). If a be the radius of the cylinder and the origin the given point, also since the plane touches (1), its equation is

$$x(x'-a) + yy' + z(z'-\gamma) + \gamma^2 - ax' - \gamma z' = 0 \text{(2) ;}$$

but, as (2) passes through the origin, $\gamma^2 - ax' - \gamma z' = 0$ (3).

Whence, suppressing accents and eliminating γ between (1) and (3), we have the required surface $(\Sigma x^2 - ax)^2 - x^2 \Sigma x^2 = 0$.

Putting $y = 0$, $x = r \cos \theta$, $z = r \sin \theta$, we get $r^2 - 2ar \sec \theta + a^2 = 0$, the Logocyclic Curve.

12543. (Professor MATZ.)—The leaf of the curve “The Devil on Two Sticks,” whose equation is $y^4 - x^4 + 100a^2x^2 - 96a^2y^2 = 0$, revolves round the axis of x ; find the expression for the volume generated.

Solution by JOHN M. COLAW, M.A.

(1) Representing the equation in an explicit form, we have

$$y^2 = 48a^2 \pm [(64a^2 - x^2)(36a^2 - x^2)]^{\frac{1}{2}}.$$

According to the formula $\pi \int (y_1^2 - y_2^2) dx$, we have for the volume

$$V = 4\pi \int_0^{6a} dx (64a^2 - x^2)^{\frac{1}{2}} (36a^2 - x^2)^{\frac{1}{2}}.$$

Putting $x = 6as$, and $\frac{36}{64} = (\frac{3}{4})^2 = \epsilon^2$, we have

$$\begin{aligned} V &= 1152\pi a^3 \int ds (1 - \epsilon^2 s^2)^{\frac{1}{2}} (1 - s^2) \\ &= 384\pi a^3 \left[s(1 - s^2)^{\frac{1}{2}} (1 - \epsilon^2 s^2)^{\frac{1}{2}} + \int \frac{2 - (1 + \epsilon^2) s^2}{(1 - s^2)^{\frac{1}{2}} (1 - \epsilon^2 s^2)^{\frac{1}{2}}} ds \right]. \end{aligned}$$

$$\begin{aligned} \text{But} \quad \int \frac{2 - (1 + \epsilon^2) s^2}{(1 - s^2)^{\frac{1}{2}} (1 - \epsilon^2 s^2)^{\frac{1}{2}}} ds &= \frac{1 + \epsilon^2}{\epsilon^2} \int \frac{(1 - \epsilon^2 s^2)^{\frac{1}{2}}}{(1 - s^2)^{\frac{1}{2}}} ds \\ &- \frac{1 - \epsilon^2}{\epsilon^2} \int \frac{ds}{(1 - s^2)^{\frac{1}{2}} (1 - \epsilon^2 s^2)^{\frac{1}{2}}} = \frac{1 + \epsilon^2}{\epsilon^2} \int (1 - \epsilon^2 \sin^2 \theta)^{\frac{1}{2}} d\theta \\ &- \frac{1 - \epsilon^2}{\epsilon^2} \int \frac{1}{(1 - \epsilon^2 \sin^2 \theta)^{\frac{1}{2}}} d\theta \quad (\text{for } s = \sin \theta) \\ &= \frac{1 + \epsilon^2}{\epsilon^2} E(\epsilon, \theta) - \frac{1 - \epsilon^2}{\epsilon^2} F(\epsilon, \theta). \end{aligned}$$

For the limits of $x = 0$, $x = 6a$, or $\theta = 0$, $\theta = \frac{1}{2}\pi$, we thus obtain

$$\begin{aligned} V &= 384\pi a^3 \left[\frac{1+\epsilon^2}{\epsilon^2} E(\epsilon, \tfrac{1}{2}\pi) - \frac{1-\epsilon^2}{\epsilon^2} F(\epsilon, \tfrac{1}{2}\pi) \right] \\ &= \tfrac{1}{2} \pi a^3 \left[25E(\tfrac{1}{2}, \tfrac{1}{2}\pi) - 7F(\tfrac{1}{2}, \tfrac{1}{2}\pi) \right]. \end{aligned}$$

$$(2) \quad V = 4\pi \int_0^{6a} (64a^2 - x^2)^{\frac{1}{2}} (36a^2 - x^2)^{\frac{1}{2}} dx.$$

Developing $(64a^2 - x^2)^{\frac{1}{2}}$ into a series by the binomial theorem, integrating, and considering that

$$\int_0^a x^{2m} (a^2 - x^2)^{\frac{1}{2}} dx = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2m-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots (2m+2)} \tfrac{1}{2} \pi a^{2m+2},$$

$$\begin{aligned} \text{we have } V &= 576\pi^2 a^3 \left[\tfrac{1}{2} - \tfrac{1}{16} \left(\tfrac{9}{16}\right) - \tfrac{1}{128} \left(\tfrac{9}{16}\right)^2 - \tfrac{5}{3072} \left(\tfrac{9}{16}\right)^3 \right. \\ &\quad \left. - \tfrac{1}{524288} \left(\tfrac{9}{16}\right)^4 - \tfrac{1}{524288} \left(\tfrac{9}{16}\right)^5 - \tfrac{1}{524288} \left(\tfrac{9}{16}\right)^6 - \dots \right] \\ &= 265 \cdot 996 \pi^2 a^3. \end{aligned}$$

13521. (G. RICHARDSON, M.A.)—The late Professor WOLSTENHOLME, M.A., Sc.D., shortly before his death, handed to me a scrap of paper, on which he had hastily scratched the following equation in tetrahedra, saying he had proved it—

$$\frac{(a+x)^2 + u'}{\sin^2 \tfrac{1}{2}(A+x)} = \frac{(a-x)^2 + u'}{\sin^2 \tfrac{1}{2}(A-x)} = \frac{(b+y)^2 + u'}{\sin^2 \tfrac{1}{2}(B+y)} = \dots = u,$$

where

$$u = 16\Delta_0 \Delta_1 \Delta_2 \Delta_3 / 9V^2,$$

$\Delta_0, \Delta_1, \Delta_2, \Delta_3$ being the areas of the triangular faces, and V the volume of the tetrahedron, and

$$u' = (\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3) (\Delta_0 + \Delta_1 - \Delta_2 - \Delta_3) (\Delta_0 + \Delta_2 - \Delta_1 - \Delta_3) (\Delta_0 + \Delta_3 - \Delta_1 - \Delta_2) / 9V^2.$$

[It would appear that, in the numerators of the equation of which a proof is now required, a, b, c are the lengths of three conterminous edges, and x, y, z of the respectively opposite edges, whilst, in the denominators, A, B, C, x, y, z are the dihedral angles respectively opposite to those edges.]

Solution by H. W. CURIEL, M.A.

Projecting three faces on to the remaining face

$$\begin{aligned} \Delta_1 \cos A + \Delta_2 \cos B + \Delta_3 \cos C &= \Delta_0, \quad \Delta_2 \cos Z + \Delta_3 \cos Y + \Delta_0 \cos A = \Delta_1, \\ \Delta_3 \cos X + \Delta_1 \cos Z + \Delta_0 \cos B &= \Delta_2, \quad \Delta_1 \cos Y + \Delta_2 \cos X + \Delta_0 \cos C = \Delta_3; \\ \therefore \Delta_1 \Delta_2 \cos Z + \Delta_2 \Delta_3 \cos X + \Delta_1 \Delta_3 \cos Y &= \tfrac{1}{2} (\Delta_1^2 + \Delta_2^2 + \Delta_3^2 - \Delta_0^2); \\ \therefore \Delta_2 \Delta_3 \cos X - \Delta_1 \Delta_0 \cos A &= \tfrac{1}{2} (\Delta_2^2 + \Delta_3^2 - \Delta_1^2 - \Delta_0^2). \end{aligned}$$

$$\text{But} \quad \sin X = \frac{3Va}{2\Delta_2\Delta_3} \quad \text{and} \quad \sin A = \frac{3Vx}{2\Delta_1\Delta_0},$$

and

$$4ax = u \sin X \sin A;$$

therefore

$$a \cot X - x \cot A = (\Delta_2^2 + \Delta_3^2 - \Delta_1^2 - \Delta_0^2) / 3V;$$

therefore, squaring, $a^2 (\operatorname{cosec}^2 X - 1) + x^2 (\operatorname{cosec}^2 A - 1) - 2ax \cot X \cot A$
 $= [(\Delta_2^2 + \Delta_3^2 - \Delta_1^2 - \Delta_0^2)/3V]^2$;

$$\therefore \frac{1}{2}(u \cos X \cos A) = -a^2 - x^2 + [4\Delta_2^2 \Delta_3^2 + 4\Delta_1^2 \Delta_0^2 - (\Delta_2^2 + \Delta_3^2 - \Delta_1^2 - \Delta_0^2)^2/9V^2]$$

$$= -(a^2 + x^2) + \frac{1}{2}u - u';$$

$$\text{therefore } u = \frac{2(a^2 + x^2) + 2u}{1 - \cos X \cos A} = \frac{4ax}{\sin X \sin A} = \frac{(a+x)^2 + u'}{\sin^2 \frac{1}{2}(X+A)}$$

$$= \frac{(a-x)^2 + u'}{\sin^2 \frac{1}{2}(X-A)} = \&c. \text{ similarly.}$$

12437. (S. TRAY, B.A.)—Find four square numbers such that the sum of every three of them shall be a square number.

Solution by the Proposer.

Let x^2, y^2, z^2, u^2 be the numbers; then $x^2 + y^2 + z^2, y^2 + z^2 + u^2, z^2 + x^2 + u^2, x^2 + y^2 + u^2$ must be squares.

$$\text{Assume } x^2 + y^2 + z^2 = (x + y + z)^2;$$

$$\text{then } xy + xz + yz = 0 \dots\dots\dots(1).$$

$$\text{Now } y^2 + z^2 + u^2 = (x + y + z)^2 - x^2 + u^2$$

$$= (2x + y + z)(y + z) + u^2 = \{u + p(y + z)\}^2;$$

Hence, putting $P = p^2 - 1, Q = q^2 - 1, R = r^2 - 1$, we have

$$2x - Py - Pz = 2pu, \quad 2y - Qz - Qx = 2qu, \quad 2z - Rx - Ry = 2ru.$$

$$\text{Let } x - P(y + z) = 0, \quad y - Q(z + x) = 0, \quad z - R(x + y) = 0;$$

$$\text{then } x = 2pu, \quad y = 2qu, \quad z = 2ru;$$

$$\text{whence we have } 1 - QR - RP - PQ - 2PQR = 0$$

$$\text{and } q^2r^2 + r^2p^2 + p^2q^2 - 2p^2q^2r^2 = 0 \dots\dots\dots(2).$$

$$\text{The condition (1) gives } qr + rp + pq = 0 \dots\dots\dots(3),$$

$$(3)^2 - (2), \text{ gives } p + q + r + pqr = 0 \dots\dots\dots(4).$$

$$\text{From (3, 4), we have } r = -\frac{pq}{p+q} = -\frac{p+q}{pq+1};$$

$$\text{therefore } q = \frac{p}{2(p^2-1)} \{1 \pm \sqrt{(4p^2-3)}\}.$$

Put $4p^2 - 3 = (2p - s)^2$; then

$$p = \frac{s^2 + 3}{4s}, \quad q = \frac{s^2 + 3}{(s+1)(s-3)}, \quad r = -\frac{s^2 + 3}{(s-1)(s+3)}.$$

We have now to make $y^2 + z^2 + 4p^2u^2, z^2 + x^2 + 4q^2u^2$, and $x^2 + y^2 + 4r^2u^2$ squares. Assume $y^2 + z^2 + 4p^2u^2 = (y + z + 2pu)^2$, &c. Then

$$x = -\frac{4pqr u}{pq + pr - qr}, \quad y = -\frac{4pqr u}{pq + qr - pr}, \quad z = -\frac{4pqr u}{pr + qr - pq}.$$

Substituting the above values of p, q, r , we have

$$x = \frac{2(s^2+3)}{(s+1)(s-3)}u, \quad y = \frac{s^2+3}{2s}u, \quad z = \frac{2(s^2+3)}{(s-1)(s+3)}u.$$

Thus we can take

$$x = (s^2-1)(s^2-9)(s^2+3), \quad y = 4s(s-1)(s+3)(s^2+3), \\ z = 4s(s+1)(s-3)(s^2+3), \quad u = 2s(s^2-1)(s^2-9).$$

Example.—Take $s = 5$; then, cancelling common factors, we have

$$x = 168, \quad y = 280, \quad z = 105, \quad u = 60.$$

Thus

$$x^2 + y^2 + z^2 = (343)^2, \quad y^2 + z^2 + u^2 = (305)^2,$$

$$x^2 + x^2 + u^2 = (207)^2, \quad x^2 + y^2 + u^2 = (332)^2.$$

The results obtained by the late Professor GILL are

$x = 4a \sin \alpha \cos \alpha$, $y = a \sin \alpha (1 - 4 \cos^2 \alpha)$, $z = a \cos \alpha (1 - 4 \sin^2 \alpha)$; which can be transformed into the proposed expressions. The same results may be found thus:—

Let the roots of the squares be $m^2 - n^2$, $cm^2 - c^{-1}n^2$, $2mn$. Then

$$x^2 + z^2 = (m^2 + n^2)^2, \quad y^2 + z^2 = (cm^2 + c^{-1}n^2)^2,$$

$$n^2 + y^2 = (1 + c^2)m^4 - 4m^2n^2 + (1 + c^{-2})n^4.$$

Let $-4m^2n^2 + (1 + c^{-2})n^4 = 0$; $\therefore (1 + c^2)n^2 = 4c^2m^2$.

Put $c = (2st)/(s^2 - t^2)$; then $(s^2 + t^2)n = 4stm$.

Take $m = s^2 + t^2$, $n = 4st$, and the roots of the squares are

$$8st(s^4 - t^4), \quad 2st\{10s^2t^2 - 3(s^4 + t^4)\}, \quad (s^2 - t^2)(s^4 + t^4 - 14s^2t^2).$$

The following results, though not so convenient, also answer the proposed conditions.

Let $x^2 + z^2 = (m + x - z)^2$, $y^2 + z^2 = (n + y - z)^2$;

therefore

$$2x = \frac{m^2 + 2mx}{m + x} = \frac{n^2 + 2ny}{n + y}.$$

Let

$$m^2 + 2mx = \lambda(n^2 + 2ny), \quad m + x = \lambda(n + y);$$

therefore

$$x = \frac{2mn - m^2 - \lambda n^2}{2(m - n)}, \quad y = \frac{m^2 + \lambda(n^2 - 2mn)}{2\lambda(m - n)};$$

$$\therefore x^2 + y^2 = \left\{ \frac{2mn - m^2 - \lambda n^2}{2(m - n)} \right\}^2 + \left\{ \frac{m^2 + \lambda(n^2 - 2mn)}{2\lambda(m - n)} \right\}^2$$

must be a square. Therefore

$n^4\lambda^4 + 2n^2(m^2 - 2mn)\lambda^3 + \{(m^2 - 2mn)^2 + (n^2 - 2mn)^2\}\lambda^2 + 2m^2(n^2 - 2mn)\lambda + m^4$ must be a square. Put the root of this $= n^2\lambda^2 + (m^2 - 2mn)\lambda + m^2$, and

we find

$$\lambda = \frac{2m^2(n^2 - n^2)}{n^2(m^2 - 4mn + 2n^2)};$$

$$\therefore x = \frac{m(2n^3 - 7mn^2 + 8m^2n - 4m^3)}{2(m - n)(2m^2 - 4mn + n^2)}, \quad y = \frac{n(-n^3 + 4mn^2 - 4m^3)}{4(m - n)(m^2 - n^2)},$$

$$z = \frac{n^3 - 2mn^2 + 2m^2n - 2m^3}{n(3n - 4m)}.$$

Example.—Take $m = 2$, $n = 5$; then $x = \frac{4}{3}$, $y = \frac{4}{3}\frac{5}{3}$, $z = \frac{1}{3}$; or, in integers, $x = 240$, $y = 275$, $z = 252$.

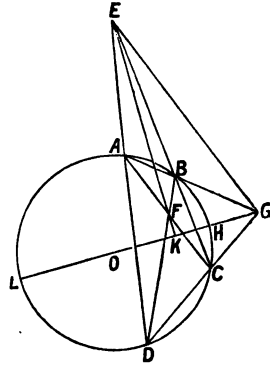
[For other solutions of the same Question, proposed (as 3549) by A. B. EVANS, M.A., see Vol. xvi., p. 108.]

13619. (W. E. JEFFARES, B.A.)—Prove that the circle described on the distance between the double points of the involution determined on any line by the sides of a quadrilateral inscribed in a circle cuts this circle orthogonally, and show that the circle on the third diagonal cuts orthogonally the same circle.

Solution by REV. J. CULLEN, B.A. ; L. E. REAY, B.A. ; and others.

The double points are harmonic conjugates of any pair of homologous points of the involution ; also the circle circumscribing the quadrilateral passes through an homologous pair. Hence the first part of the question, since the circle described on the distance between two points cuts orthogonally any circle passing through their harmonic conjugates.

Now, by a known theorem, EK is the polar of G, and is therefore perpendicular to the diameter through G. Hence the circle described on EG passes through K ; also (LKHG) is harmonic. Therefore, &c.



4680. (Professor BURNSIDE, M.A.)—(1) Determine the degree of the locus represented by the equation

$$\frac{a^2}{p^2} + \frac{\beta^2}{p_1^2} + \frac{\gamma^2}{p_2^2} - 2(\beta\gamma/p_1 p_2) \cos A - 2(\gamma\alpha/p_2 p) \cos B - 2(\alpha\beta/pp_1) \cos C = 1,$$

where α, β, γ are the distances of any point from the vertices of the triangle ABC, and p, p_1, p_2 the perpendiculars of the same triangle.

(2) Show that the curve given by the above equation is the same as that referred to by Professor CAYLEY in Quest. 2110, which is as follows:—Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity.

Solution by Professor SANJANA.

(1) If the sides of the pedal triangle be d, e, f , corresponding to α, β, γ respectively, we have $\alpha/p = (d \operatorname{cosec} A)/p = R\bar{d}/\Delta$, where R and Δ have the usual meanings with regard to the triangle. Hence the equation given becomes $d^2 + e^2 + f^2 - 2ef \cos A - 2fd \cos B - 2de \cos C = \Delta^2/\bar{R}^2$,

which, since

$$d = (y^2 + z^2 + 2yz \cos A)^{\frac{1}{2}}, \text{ \&c.,}$$

will, when rationalized, give an equation of the eighth degree in trilinears. If the angles of the pedal triangle be D, E, F respectively, the equation can also be written thus : $\Sigma ef \cos A = \Sigma ef \cos D$.

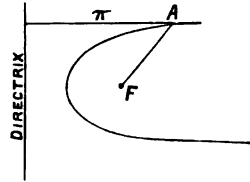
(2) This does not readily follow. It would seem that the equation of the eighth degree has a cubic factor.

[The PROPOSER makes the following remarks:—"There is an identical relation connecting the perpendiculars from A, B, C to the directrix (any line what ever), or connecting FA, FB, FC or α, β, γ in question. The identical relation in general is

$$(1, 1, 1 - \cos A - \cos B - \cos C)$$

$$(\pi/p, \pi_1/p_1, \pi_2/p_2)^2 = 1,$$

π, π_1, π_2 the perpendiculars from A, B, C on the line.]



13552. (Professor A. DROZ-FARNY.)—Soient Δ et Δ' deux cercles de centres O et O' et se coupant en A et B. Une transversale variable menée par A coupe Δ en α et Δ' en α' . La parallèle menée par α à O'A coupe Δ en β , et la parallèle menée par α' à O'A coupe Δ' en β' : (1) Les trois points A, β, β' sont en ligne droite; (2) si les droites $\alpha\beta$ et $\alpha'\beta'$ se coupent en P, quel est le lieu de ce point?

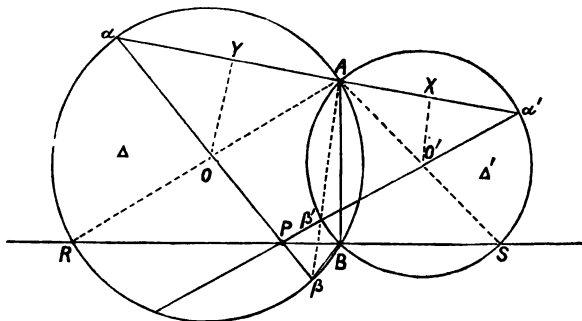
Solution by F. H. PRACHELL, B.A.; W. J. DOBBS, M.A.; and others.

1. By parallels $\angle \alpha'\beta'A = \angle O\alpha\beta' = \frac{1}{2}\pi - \angle AR\beta$ (for $A\beta R$ is in semicircle)
 $= \frac{1}{2}\pi - \angle A\alpha\beta = \frac{1}{2}\pi - \angle O'A\alpha'$ (by parallels)

(O'X is drawn perpendicular from O')

$$= \angle AO'X = \frac{1}{2}\angle AO'\alpha';$$

therefore β' lies on circle Δ' .



2. Now, as $\alpha A B \beta$ is a quadrilateral in a circle, therefore

$$\angle A B \beta = \pi - \angle \alpha,$$

and $\angle AB\beta' = \angle \alpha'$ in same segment; therefore

$$\beta B\beta' = \pi - \alpha - \alpha' = \pi - \beta P\beta';$$

therefore $P\beta'B\beta$ is concyclic; therefore

$$\angle P\beta B\beta' = \angle P\beta\beta' = \frac{1}{2}\angle AOB = \angle YOA = \frac{1}{2}\pi - \angle OAY = \frac{1}{2}\pi - \alpha'.$$

Also $\angle AB\beta' = \alpha'$; therefore $\angle ABP = \frac{1}{2}\pi$.

So locus of P is a straight line through B at right angles to AB, and it also passes through R, S, the points where AO, AO' cut the circles.

13539. (MORGAN BRIERLEY).—Given the vertical angle, the sum of the sides including it, and the radius of the inscribed circle, to construct the triangle.

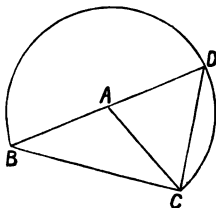
Solution by Rev. T. ROACH, M.A.; W. E. JEFFARES, B.A.; and others.

Since $r/\tan \frac{1}{2}A = s - a = \frac{1}{2}(b + c - a)$, therefore a is known, and the problem reduces to the well-known geometrical one—"Given base, sum of sides, and vertical angle, construct the triangle."

On BC, or a , as base, describe a segment containing angle $\frac{1}{2}A$.

Let radius BD = $b + c$, and make $\angle DCA = \angle BDC$.

Then BAC is the triangle.



13511. (R. CHARTRES).—A circle passes through FERMAT's point of an isosceles triangle and touches the base; find (1) the locus of its centre, and (2) the mean value of the circles.

Solution by the PROPOSER.

Let F be FERMAT's point,

$$FD = p = \frac{1}{2}(a\sqrt{3}),$$

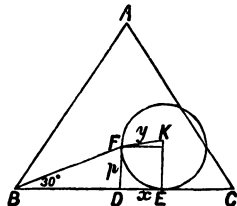
(x, y) the coordinates of centre K.

(1) Then $y^2 = x^2 + (y - p)^2$;

therefore the locus is a parabola.

(2) The mean value of the circle

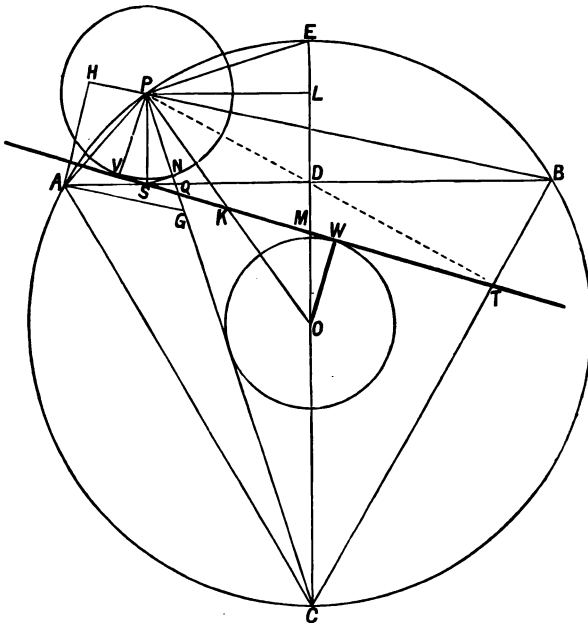
$$\begin{aligned} &= \frac{2\pi}{a} \int_0^a y^2 \cdot dx = \frac{\pi}{2ap^2} \int_0^a (x^2 + p^2)^2 \cdot dx \\ &= \frac{\pi a^2}{10}. \end{aligned}$$



13639. (D. BIDDLE.)—If P be a point on the circumcircle of ABC , an equilateral triangle of side a , and if l be its perpendicular distance from the SIMSON-line drawn in respect of it (in the case of an equilateral triangle, any such SIMSON-line is equidistant from P and O the centre), prove that PA, PB, PC represent in length the three roots of the cubic equation $x^3 + qx + r = 0$, when $q = -a^2$ and $r = \pm 4R^2l = \pm \frac{4}{3}a^2l$. [All cases irreducible by CARDAN's method—that is, all cases having three real roots—are included herein; for, if the value of q be supposed fixed, r can range from 0 to $\pm 2(-\frac{1}{3}q)^{\frac{1}{2}}$, whilst the three roots of the equation respectively range from 0 to $(-\frac{1}{3}q)^{\frac{1}{2}}$, from $(-q)^{\frac{1}{2}}$ to $(-\frac{1}{3}q)^{\frac{1}{2}}$, and from $(-q)^{\frac{1}{2}}$ to $2(-\frac{1}{3}q)^{\frac{1}{2}}$; two of them being of one sign, and the third of the opposite. But the finding of P , when a and l alone are given, would appear to involve the trisection of an angle.]

Solutions (1) by the PROPOSER, (2) by PROFESSOR A. DROZ-FARNY.

(1) Let x_1, x_2, x_3 be the three roots of the given equation. Then we have $x_1 + x_2 + x_3 = 0$, whence $x_3 = -(x_1 + x_2)$; $x_1x_2 + x_1x_3 + x_2x_3 = q$, whence



$x_3 = \pm(-q - \frac{1}{3}x_1^2)^{\frac{1}{2}} - \frac{1}{2}x_1$; and $x_1x_2x_3 = -r$. Therefore, disregarding the signs of the roots, and taking $AB^2 = -g$, we have to prove

- (i.) that $PA + PB = PC$;
 (ii.) that $PB = (AB^2 - \frac{1}{3}PA^2)^{\frac{1}{2}} - \frac{1}{3}PA$;
 (iii.) that $PA \cdot PB \cdot PC = \frac{4}{3}AB^2 \cdot l$.

(i.) $\angle APC = BPC = \frac{1}{3}\pi$. Make $PG = PA$, and join AG . Then APG is an equilateral triangle, and triangle AGC is in all respects equal to APB , having $\angle AGC = APB$, $\angle ACG = ABP$, and $AC = AB$. Therefore $PC = PA + PB$.

(ii.) Produce BP to H , and draw AH perpendicular to it. Then APH is half an equilateral triangle, having $AH = \frac{1}{2}\sqrt{3}PA$, and $PH = \frac{1}{2}PA$; and it is easy to see that $PB = (AB^2 - \frac{1}{3}PA^2)^{\frac{1}{2}} - \frac{1}{3}PA$. Moreover, it is now clear that, if the value of $q (= -AB^2)$ remain fixed, the three roots of the equation may respectively range from 0 to AE [or 0 to $(-\frac{1}{3}q)^{\frac{1}{2}}$], from AB to EB [or $(-q)^{\frac{1}{2}}$ to $(-\frac{1}{3}q)^{\frac{1}{2}}$], and from AC to EC [or $(-q)^{\frac{1}{2}}$ to $2(-\frac{1}{3}q)^{\frac{1}{2}}$]. The ranges given within brackets are dependent upon the fact that when $r = 0$ the equation reduces to $x^2 + q = 0$, whence $x = \pm(-q)^{\frac{1}{2}}$; also upon the fact that $dr/dx = -d(x^2 + qx)/dx = -(3x^2 + q)$, which reaches zero when $x = \pm(-\frac{1}{3}q)^{\frac{1}{2}}$. Thus, taking the values in a neutral sense, we have

$$0 < x_1 < (-\frac{1}{3}q)^{\frac{1}{2}} < x_2 < (-q)^{\frac{1}{2}} < x_3 < 2(-\frac{1}{3}q)^{\frac{1}{2}}.$$

The corresponding values of r at the several limits are 0, $\pm 2(-\frac{1}{3}q)^{\frac{1}{2}}$, 0, $\mp 2(-\frac{1}{3}q)^{\frac{1}{2}}$. Between these are all the values with which CARDAN's method fails to deal.

(iii.) In order to prove that $PA \cdot PB \cdot PC = \frac{4}{3}AB^2 \cdot l$, where l is the distance of P from its SIMSON-line, let PC cross AB at Q . Then we have the similar triangles APQ , CPB , and $PA : PQ = PC : PB$. Therefore $PA \cdot PB \cdot PC = PQ \cdot PC^2$. Next, draw PS perpendicular to AB , and SN perpendicular to PC . Then PCE , PQS , PSN are similar triangles, and $PQ \cdot PC^2 = PN \cdot EC^2 = \frac{4}{3}AB^2 \cdot PN$. Again, T being the foot of the perpendicular from P on BC , ST is the SIMSON-line referred to and P , S , T , B are concyclic. Therefore $\angle SPT = SBT = BPC$ and $\angle SPC = BPT = BST = ASV = SPV$ (PV being perpendicular to the SIMSON-line, and $\equiv l$). It is easy therefore to see that $PN = l$, and $r = \frac{4}{3}AB^2 \cdot l = PA \cdot PB \cdot PC$. Moreover, EP being perpendicular to PC , and PL to EC , $\angle EPL = SPN = BST$, and triangle DSM is equal in all respects to ELP . But $OD = DE$; therefore $OM = LD = PS$, and OW drawn perpendicular to the SIMSON-line, from the centre, equals PV . Consequently, the SIMSON-line bisects the radius OP in K ; and, as we have the values of OP and PV given by the equation, we can readily find the angle VPK ($= KOW$) by drawing two circles, of radius $= PV$, from centres whose distance apart $= OP$, and drawing an internal tangent corresponding to VW . This angle is trisected by the lines PS , PC . Similarly, $\angle MOW = \frac{1}{3}KOW$, and, if we place AB (as in the figure) perpendicular to OM , P is at once found. Thus, the future discoverer of a means of geometrically trisecting a given angle will effect great things, this of solving cubic equations, irreducible by CARDAN's method, amongst the number.

It is further interesting to note that $\triangle ABC : \triangle ABP = qx_3 : r$; and that, if perpendiculars be drawn from C on PA , PB , the distance (equal) of their feet from A , B respectively represents the value of that portion of the two roots which is usually expressed with the radical sign.

(2) Supposons P situé sur l'arc BC : c'est un fait bien connu qu'alors

$$PA = PB + PC,$$

et par conséquent

$$(-PA) + PB + PC = 0.$$

Dans ce cas PA devra être pris négativement. Posons

$$\text{arc BP} = 2\alpha;$$

on aura

$$PB = 2R \sin \alpha;$$

$$PA = 2R \sin (\alpha + 60);$$

$$PC = 2R \sin (60 - \alpha).$$

Soient D et E les pieds des perpendiculaires abaissées de P sur BC et AC; le quadrilatère PCED est inscrit dans une circonférence de diamètre PC. Dans cette circonférence, comme l'arc ED est de 120° , on a

$$EP = CP \sin (60 + \alpha), \quad PD = CP \sin \alpha, \quad ED = CP \sin 60.$$

En exprimant de deux manières la surface du triangle PED, on a

$$PE \cdot PD \sin 60 = ED \cdot l;$$

$$(CP)^2 \sin (60 + \alpha) \sin \alpha \sin 60 = l \cdot CP \sin 60;$$

$$l = 2R \sin \alpha \cdot \sin (60 + \alpha) \sin (60 - \alpha);$$

$$4R^2 l = PA \cdot PB \cdot PC; \quad r = 4R^2 l = \frac{4}{3}a^2 l.$$

On a encore

$$\begin{aligned} -PA(PB + PC) + PB \cdot PC &= 4R^2 \{ \sin \alpha \sin (60 - \alpha) - \sin (\alpha + 60) \\ &\quad [\sin \alpha + \sin (60 - \alpha)] \} \\ &= 4R^2 \left(-\frac{3}{4} \right) = -3R^2 = -a^2, \\ q &= -a^2. \end{aligned}$$

Note on Question 4329. Tests for higher powers of 2. By Lt.-Col. ALLAN CUNNINGHAM, R.E.

I have compared Mr. K. S. PUTNAM's newly computed value of 2^{721} with Mr. SHANKS's (published in 1853), and find them in exact agreement. This is the best test possible.

All the various "tests" proposed—viz., division of $(2^{721} - 1)$ by $127 = (2^7 - 1)$, and by $(2^{103} - 1)$, or of $(2^{720} - 1)$ by its numerous known small divisors—besides being merely *negative* tests, have, with the exception of "casting out the nines," the disadvantage of involving the labour of *actual division* of these large numbers. The following "tests," analogous to "casting out the nines," are much simpler, in not involving heavy divisions.

The number, being printed in periods of three figures, may be looked on as printed in the scale of 1000. By simply adding each column as it

stands, and then adding the sum of the columns, a certain number is obtained, far less than 2^{721} . This result should be written in periods of three digits, and treated in the same way. By repeating this process, a number will finally result less than 999, and is, in fact, the actual residue to modulus 999. This residue should be divided by 27 and 37 (the factors of 999) in turn, so as to yield the actual remainders to moduli 27 and 37.

Now $2^{18} \equiv 1 \pmod{27}$ and $2^{36} \equiv 1 \pmod{37}$;

so that the remainders should be 2 to both moduli.

Again, by adding six ciphers to the number as printed, it will be seen to be arranged in lines of 45 figures each, so that, by ruling two columns, it may be looked on as printed in three columns with 15 figures in each row, and may therefore be considered as printed in the scale of 10^{15} .

By simply adding each such column, and then adding the sum of the columns, a certain number is obtained. After pointing into periods of 15 figures each (from the right), this number should be similarly treated until a result is obtained less than $(10^{15} - 1)$, which is, in fact, the remainder to modulus $(10^{15} - 1)$. This remainder should then be divided by 31, 41, 271 (the small factors of $10^{15} - 1$), in turn, so as to obtain the actual remainders to those moduli.

Now $2^6 \equiv 1 \pmod{31}$, $2^{20} \equiv 1 \pmod{41}$, $2^{135} \equiv 1 \pmod{271}$;

so that the remainders should be

$2 \pmod{31}$, $2 \pmod{41}$, and $56 \pmod{271}$.

The tests for casting out the *elevens* and the factors of $(10^3 + 1)$, $(10^{15} + 1)$ are equally simple, but the mode of printing the number in rows of 45 figures does not lend itself conveniently to the preliminary additions.

Additional Note to Quest. 4329 by H. J. WOODALL, A.R.C.S.

In addition to the factors of $2^{720} - 1$ given on page 27 of the present volume (from which 6168 should be erased), the following also divide it, viz.,

$181 \times 577 \times 188,37001 \times 42782,55361 \times 48,78248,87233 \times 157,94029,41661$; of these the former three are prime, the latter three (if not prime) have factors of the forms $80l + 1$, $144m + 1$, $180n + 1$, respectively.

Note on the Factorization of $(2^{720} - 1)$. By Lt.-Col. ALLAN CUNNINGHAM, R.E.

The number $N = (2^{720} - 1)$ may be more completely resolved into its *prime* factors than in the results given in the solution of Quest. 4329, on page 26, Vol. LXVIII. The simplest plan (to avoid missing any small factors) appears to be to resolve N into its obvious algebraic factors, and then resolve them separately.

Thus $N = (2^{45} - 1)(2^{45} + 1)(2^{90} + 1)(2^{180} + 1)(2^{360} + 1)$.

Now, by help of the tables on pages 4, 5 of ED. LUCAS's paper "Sur la Série récurrente de FERMAT," Rome, 1879, there may be deduced

$$(2^{45} - 1) = 7 \cdot 31 \cdot 73 \cdot 151 \cdot 631 \cdot 23,311,$$

$$(2^{45} + 1) = 27 \cdot 11 \cdot 19 \cdot 331 \cdot 18,837,001,$$

$$(2^{90} + 1) = (25 \cdot 109 \cdot 181 \cdot 1321 \cdot 54,001) \cdot (13 \cdot 37 \cdot 41 \cdot 61 \cdot 29,247,661).$$

The two largest factors, $(2^{180} + 1)$, $(2^{360} + 1)$, have probably never been completely factorized, but may be partially factorized as follows:—

$$\text{Since} \quad (x^{15} + 1) = \frac{(x^5 + 1)(x^3 + 1)}{x + 1} \cdot F(x),$$

$$\text{where} \quad F(x) = (x^8 + x^7 - x^5 - x^4 - x^3 + x + 1).$$

Making $x = 2^{12}$ and 2^{24} in turn, and resolving $(x^5 + 1)$, $(x^3 + 1)$ by LUCAS's table,

$$\begin{aligned} (2^{180} + 1) &= \frac{2^{60} + 1}{2^{12} + 1} \cdot (2^{36} + 1) \cdot F(2^{12}) \\ &= (61,681 \cdot 4,562,284,561) (17 \cdot 241 \cdot 433 \cdot 38,737) \cdot F(2^{12}). \end{aligned}$$

$$\begin{aligned} \text{Also} \quad (2^{360} + 1) &= \frac{(2^{120} + 1)(2^{72} + 1)}{(2^{24} + 1)} \cdot F(2^{24}) \\ &= (577 \cdot 487,824,887,233) \cdot (2^{120} + 1) \cdot F(2^{24}); \end{aligned}$$

and making $x = 2^8$, and reducing again by LUCAS's table,

$$\begin{aligned} (2^{120} + 1) &= \frac{(2^{40} + 1)(2^{24} + 1)}{(2^8 + 1)} \cdot F(2^8) \\ &= (4,278,255,361) \cdot (97 \cdot 257 \cdot 673) \cdot F(2^8). \end{aligned}$$

All the large factors above are quoted as *primes* in LUCAS's table except the three largest, viz., $F(2^8)$, $F(2^{12})$, $F(2^{24})$, whose prime or composite character is believed to be unknown. Their values are easily obtained by substituting $x = 2^8$, 2^{12} , 2^{24} in the formula for $F(x)$;

thus $F(2^8) = 18,518,800,563,924,107,521$; and so on.

13643. (R. KNOWLES, B.A.)—Prove that

$$1 + (n+1)x + (n+2)(x^2/1.2) + (n+3)(x^3/1.2.3) + \dots = (x+n)\epsilon^x - (n-1).$$

Solution by G. M. GRACE, B.A., B.Sc.; Professors SAVAGE and A. DROZ-FARNY; and many others.

The given expression

$$\begin{aligned} &= n [1 + x + (x^2/2!) + \dots - 1] + 1 + x + x^2 + (x^3/2!) + \dots \\ &= n(\epsilon^x - 1) + 1 + x\epsilon^x = \epsilon^x(n+x) - (n-1). \end{aligned}$$

[The PROPOSER makes the following remarks:—"The symbolical representation of the series in the question is $O + (D+n)\epsilon^x$, where O must be determined by inspection. In solving Quest. 13375, pro-

posed by Professor SYLVESTER (Vol. LXVII., p. 57), it appeared to me that the form $(D+1)^n$ is applicable to other series besides the development of $(1-x)^{-1}$. Although, perhaps, not the readiest form for solving any particular question, it is useful for purposes of investigation.]

13648. (R. F. DAVIS, M.A.)—Resolve 1,000,000,000,004 into factors.

Solutions by (1) JOSEPH BLATER; (2) Lt.-Col. ALLAN CUNNINGHAM, R.E.; C. E. BICKMORE, M.A.; and H. W. CURJEL, M.A.

(1) Using the "Table of Quarter-Squares of all Whole Numbers from 1 to 200,000," by JOSEPH BLATER (London: Trübner, 1888), results are most readily reached when the two factors to be found are not very far removed from the square root of the given number.

The number in question divided by 4

	$= 250,000,000,001$	to which the nearest quarter-square is	$250,001,000,001$	with argument 1,000,002
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and the remainder	1,000,000	has for argument	2,000	
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and then the half-sum	= 1,002,002 : 2
half-difference	= 998,002 : 2 ;

501,001 is a prime,

$499,001 = 17 \times 149 \times 197$;

therefore $1,000,000,000,004 = 2 \times 2 \times 17 \times 149 \times 197 \times 501,001$.

(2) Here $N = 1,000^4 + 4 \cdot 1^4$

$$= (1,000^2 - 2 \cdot 1,000 + 2)(1,000^2 + 2 \cdot 1,000 + 2) \\ = [2 \cdot 17 \cdot 149 \cdot 197][2 \cdot 501,001].$$

Any number of form $N = (x^4 + 4y^4)$ is at once resolved into two factors as

$$N = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2),$$

a result ascribed to Mr. AURIFEUILLE (see LUCAS, *Théorie des Nombres*, Art. 73).

Addendum.—Similarly may be factorized

$$N = 4,000,000,000,001 = 1^4 + 4 \cdot 1,000^4 \\ = (1 - 2 \cdot 1,000 + 2 \cdot 1,000^2)(1 + 2 \cdot 1,000 + 2 \cdot 1,000^2) \\ = [277 \cdot 7,213] \cdot [2,002,001].$$

13606. (Professor NEUBERG.)—On considère les paraboles qui touchent une droite donnée a en un point donné A et dont la tangente au sommet passe par un point donné B . Trouver l'enveloppe de l'axe.

4757. (W. S. B. WOOLHOUSE, F.R.A.S.)—Innumerable pairs of points are taken at random on the area of a given circle, their distance asunder not exceeding the radius of the circle; determine the law of the distribution of the collective mass of points.

Solution by H. W. CURJEL, M.A.

Let O be the centre of the given circle ABC (radius a). The density at P is clearly proportional to the area common to circle ABC and an equal circle with centre P; i.e., to

$$2a^2 \cos^{-1} r/2a - r(a^2 - \frac{1}{4}r^2)^{\frac{1}{2}}, \quad \text{where } OP = r.$$

3867. (J. B. SANDERS.)—What number of degrees must a given volume of air be heated to double its elasticity?

Solution by H. J. WOODALL, A.R.C.S.

A gas is perfectly elastic; for gases we have $pv = Rt$; therefore, if v is constant, $p' = 2p$, when $t' = 2t$; t, t' being absolute temperatures measured by the air thermometer.

12754. (Professor PRENTISS.)—Find the coordinates of the forty-nine points of intersection, real and imaginary, of the two curves

$$(x^5 + 1)y = (y^2 + 1)x^3 \quad \text{and} \quad (y^6 + 1)x = 9(x^2 + 1)y^3.$$

Solution by Rev. J. L. KITCHIN, M.A.; Prof. BHATTACHARYA; and others.

The equations are

$$x^5 + 1/x^3 = y + 1/y, \quad 9(x + 1/x) = y^3 + 1/y^3 \dots\dots\dots(1, 2).$$

From (2) + 3 (1), $y^3 + 1/y^3 + 3(y + 1/y) = 3[x^3 + y^3 + 3(x + 1/x)]$;

therefore $y + 1/y = (3^{\frac{1}{3}}, \text{ or } 3^{\frac{1}{3}}\omega, 3^{\frac{1}{3}}\omega^2)(x + 1/x).$

This and (1) give $x + 1/x = 0$; therefore $x = \pm i$, and y twelve values, and

$$x^2 - 1 + 1/x^2 = 3^{\frac{1}{3}}, \text{ \&c., } x + 1/x = \pm(3 + 3^{\frac{1}{3}})^{\frac{1}{2}}, \pm(3 + 3^{\frac{1}{3}}\omega)^{\frac{1}{2}}, \pm(3 + 3^{\frac{1}{3}}\omega^2)^{\frac{1}{2}};$$

therefore $y^3 + 1/y^3 = \pm 9(3 + 3^{\frac{1}{3}})^{\frac{1}{2}}, \pm 9(3 + 3^{\frac{1}{3}}\omega)^{\frac{1}{2}}, \pm(3 + 3^{\frac{1}{3}}\omega^2)^{\frac{1}{2}}.$

Hence y has 36 values, corresponding to $x + 1/x = 0$, (2) and obviously $x = 0, y = 0$ satisfy the equations. Hence there are 49 points of intersection, and the above equations give the values of y and x .

$-m \cos n \sin (m \sin n)$, that of $\sin (m \cos n)$ is $-m \sin n \cos (m \cos n)$, and that of $\cos (m \cos n)$ is $m \sin n \sin (m \cos n)$.

Solution by R. KNOWLES, M.A.; BELLE EASTON; and others.

(1) m variable, n constant: $u = \sin (m \sin n)$; put $x = m \sin n$,
 $dx/dm = \sin n$, and $du/dx = \cos x = \cos (m \sin n)$;
 therefore $du/dm = dx/dm \cdot du/dx = \sin n \cos (m \sin n)$,
 and similarly of the others.

(2) m constant, n variable: $u = \sin (m \sin n)$; $x = m \sin n$,
 $dx/dn = m \cos n$, $du/dx = \cos (m \sin n)$;
 therefore $du/dn = m \cos n \cos (m \sin n)$,
 and so of the others.

7767. (D. EDWARDS.)—If K, E be the complete elliptic integrals of the first and second order, $\Delta = (1 - k^2 \sin^2 \phi)^{1/2}$, $k_1^2 = 1 - k^2$, and m be a positive integer, prove that

$$\int_0^{2K} \frac{u}{\operatorname{dn}^2 u} du = \frac{2}{k_1^2} KE, \quad \int_0^{2K} \frac{u}{\operatorname{dn}^3 u} du = \frac{\pi}{2k_1^3} (1 + k_1^2) K,$$

$$\int_0^{2K} \frac{u}{(\operatorname{dn} u)^{2m}} du = \frac{2K}{k^{2m}} \int_0^{1/2\pi} \Delta^{2m-1} d\phi.$$

Solution by H. W. CURJEL, M.A.

Since

$$\operatorname{dn} (2K - u) = \operatorname{dn} u,$$

$$\int_0^{2K} \frac{u du}{\operatorname{dn}^2 u} = \int_0^K \frac{u + 2K - u}{\operatorname{dn}^2 u} du = 2K \int_0^K \frac{du}{\operatorname{dn}^2 u} = \frac{2K}{k_1^2} \int_0^K \operatorname{dn}^2 u du = 2KE/k_1^2.$$

$$\begin{aligned} \int_0^{2K} \frac{u du}{\operatorname{dn}^3 u} &= 2K \int_0^K \frac{du}{\operatorname{dn}^3 u} = \frac{K(1 + k_1^2)}{k_1^2} \int_0^K \frac{du}{\operatorname{dn} u} \\ &= \frac{K(1 + k_1^2)}{k_1^2} \int_0^K \frac{d(\operatorname{sn} u)}{\operatorname{cn} u (1 - k^2 \operatorname{sn}^2 u)} = \frac{K(1 + k_1^2)}{k_1^2} \int_0^1 \frac{2(1 + z^2) dz}{(1 + z^2)^2 - 4k^2 z^2} \\ &\quad \left(\text{where } \operatorname{sn} u = \frac{2z}{1 + z^2} \right) \end{aligned}$$

$$= \frac{K(1 + k_1^2)}{k_1^2} \times \frac{1}{k_1} \times \frac{1}{0} \left\{ \tan^{-1} \frac{2zk_1}{1 - z^2} \right\} = \frac{\pi}{2k_1^3} (1 + k_1^2) K,$$

$$\int_0^{2K} \frac{u du}{(\operatorname{dn} u)^{2m}} = 2K \int_0^K \frac{du}{(\operatorname{dn} u)^{2m}}.$$

Let $\operatorname{dn} u = k_1/\Delta$; then $\phi = \frac{1}{2}\pi$ and 0, when $u = 0$ and K , and

$$\operatorname{sn} u = (\cos \phi)/\Delta, \quad \operatorname{cn} u = (k_1 \sin \phi)/\Delta,$$

and

$$-k^2 \operatorname{sn} n \operatorname{cn} u du = (k_1 k^2 \sin \phi \cos \phi d\phi)/\Delta^3;$$

therefore

$$du = -d\phi/\Delta;$$

therefore

$$\int_0^{2K} \frac{u du}{(dn u)^{2m}} = \frac{2K}{k_1^{2m}} \int_0^{1\pi} \Delta^{2m-1} d\phi.$$

6474. (Professor CHASE, LL.D.)—If f represent any force whatever that varies inversely as the square of the distance, and r be the radius of a perpetual circular oscillation produced by the force, find the mean velocity of a synchronous radial oscillation.

Solution by H. J. WOODALL, A.R.C.S.

The time of circular oscillation = $2\pi (r/f)^{1/2}$.

A radial oscillation describes $4r$ in length.

Mean velocity required = $4r/\{2\pi (r/f)^{1/2}\} = 2 (rf)^{1/2}/\pi$.

Note on Question 13484. By R. TUCKER, M.A.

This Question, now appearing in Mr. JEFFARES' name and solved by Mr. COONEY in Vol. LXVII., p. 110, was proposed by me in the *Educational Times*, Feb. 1865, as Question 1649, and Solutions of it appear in the *Reprint*, Vol. III., p. 58. Placing the two Questions side by side, they are as follows:—

1649. (R. TUCKER, M.A.)—From the ends of a diameter of a given circle perpendiculars are drawn on the sides of an inscribed triangle; prove that the two feet-perpendicular lines intersect at right angles on the nine-point circle of the triangle.

13484. (W.E. JEFFARES, B.A.)—If ABC is a triangle, and O the circumcentre, through O draw any diameter LOM intersecting the circumcircle in L and M; find the locus of the intersection of the SIMSON-lines corresponding to L and M.

Clearly "feet-perpendicular line" = modern "SIMSON-line." A note by ARCHER STANLEY to his solution of my question (*Reprint*, Vol. III., p. 58) draws attention to a paper by STEINER in Vol. LIII. of *Crelle's Journal*. From an abridgment of this paper given on pp. 97–99 of the same volume of the *Reprint*, it appears that I had been anticipated by that great geometer. (SALMON says: "Pereant qui nostra ante nos dixerunt.") It was repropoed in the *Nouvelles Annales*, August, 1884 (p. 397, Quest. 1473), by M. N. GOFFART. I pointed out to the editor my priority, at any rate to GOFFART, and he put in a note, "on m'a écrit de Londres." &c., with questionable courtesy. Dr. CASEY put the same question in the first four editions of his *Sequel* (though I referred him to the *Educational Times Reprint*) as due to Mr. R. GRAHAM. There is a good account of the SIMSON (i.e., WALLACE) line in the *Proceedings of the Edinburgh Mathematical Society*, Vol. III., pp. 77–93.

APPENDIX.

UNSOLVED QUESTIONS.

5528. (Col. A. R. Clarke, C.B., F.R.S.)—P is a point on the earth's surface, supposed spherical; Q a point vertically below it—in the earth's mass; Q' a point vertically above P, so that $PQ' = PQ$. Imagine the sphere of matter which has Q for centre and QP for radius to be removed, and transferred to Q' as its centre. Determine the form of the surface of the sea as thus disturbed; QP being small in comparison with the radius of the earth, and the density of the removed sphere being supposed uniform.

5530. (J. W. L. Glaisher, F.R.S.)—There being $(n+1)^2$ quantities

$$\begin{array}{ccccccc} a_1a_1, & a_1a_2, & \dots & a_1a_n, & a_1a_{n+1}, \\ a_2a_1, & a_2a_2, & \dots & a_2a_n, & a_2a_{n+1}, \\ \dots & \dots & \dots & \dots & \dots \\ a_n, & a_na_2, & \dots & a_na_n, & a_na_{n+1}, \\ a_{n+1}a_1, & a_{n+1}a_2, & \dots & a_{n+1}a_n, & a_{n+1}a_{n+1}, \end{array}$$

of which only $\frac{1}{2}(n+1)(n+2)$ are different, as it is supposed that $a_p a_q = a_q a_p$, then prove that

$$\begin{vmatrix} a_1a_1 & a_1a_2 & \dots & a_1a_n & a_1a_{n+1} \\ a_2a_1 & a_2a_2 & \dots & a_2a_n & a_2a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & \dots & a_na_n & a_na_{n+1} \\ a_{n+1}a_1 & a_{n+1}a_2 & \dots & a_{n+1}a_n & a_{n+1}a_{n+1} \end{vmatrix} + \begin{vmatrix} a_1a_1 & a_1a_2 & \dots & a_1a_n \\ a_2a_1 & a_2a_2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & \dots & a_na_n \end{vmatrix} \\ = a_{n+1}a_{n+1} - \frac{(a_1a_{n+1})^2}{a_1a_2} - \frac{(a_2a_{n+1} \cdot 1)^2}{a_2a_3 \cdot 1} - \frac{(a_3a_{n+1} \cdot 2)^2}{a_3a_4 \cdot 2} \dots - \frac{\{a_na_{n+1}(n-1)\}^2}{a_na_n(n-1)}$$

where $a_2a_{n+1} \cdot 1 = a_2a_{n+1} - \frac{a_2a_1 \cdot a_{n+1}a_1}{a_1a_1}$, $a_2a_2 \cdot 1 = a_2a_2 - \frac{a_2a_1 \cdot a_2a_1}{a_1a_1}$;

and, generally, $a^pa_q \cdot i = a_pa_q \cdot i - 1 - \frac{\{a_pa_i \cdot (i-1)\} \{a_qa_i \cdot (i-1)\}}{a_ia_i \cdot i - 1}$.

5532. (Elizabeth Blackwood.)—Find the chance that n random chords will give at least m intersections within a given circle.

5537. (Prof. Sylvester, F.R.S.)—A *prepared* quantic being defined as one in which the numerical multipliers are the *square roots* of the corresponding binomial or polynomial numbers, so that, for example,

$$\sum ax^3 + \sqrt{3} \cdot \sum bx^2y + \sqrt{6} \cdot \sum cxyz$$

is a prepared ternary cubic, prove that in any prepared quantic reciprocal substitutions impressed on the variables $(x, y, \dots z)$ induce reciprocal substitutions on the elements $(a, b, c, \dots l)$. *E.g.*—Taking the

prepared binary quadratic $Q = ax^2 + \sqrt{2} \cdot bxy + cy^2$, let $\begin{vmatrix} f & g \\ h & k \end{vmatrix}$ be any matrix of substitution; its reciprocal is $\begin{vmatrix} k & -h \\ -g & f \end{vmatrix}$. On writing $fx + gy$ for x , and $hx + ky$ for y , Q will become $Ax^2 + \sqrt{2} \cdot Bxy + Cy^2$; and on writing $kx - hy$ for x , and $-gx + fy$ for y , Q will become $A'x + \sqrt{2} \cdot B'xy + C'y^2$.

A, B, C will be binary functions of a, b, c , and the relations between these two sets of letters may be expressed in the usual manner by a matrix of the third order. So, again, A', B', C' will be connected with a, b, c by another matrix of the same order. By virtue of the above theorem these two matrices will be reciprocal to one another.

5541. (Professor Crofton, F.R.S.)—If ρ denote the distance of two points taken at random within any convex area Ω , and if C be the length of a chord of the area, whose coordinates are p, ω , prove the following expressions for the mean values:—

$$M\left(\frac{1}{\rho}\right) = \frac{1}{\Omega^2} \iint C^2 dp d\omega, \quad M(\rho) = \frac{1}{6\Omega^2} \iint C^4 dp d\omega,$$

the integrals extending to all positions of the chord C .

5547. (Christine Ladd.)— O, O' and M, M' are reciprocal points with respect to a circle whose centre is C , and OM is equal to MO' . At any point T draw a tangent to the circle, and let it cut OP , perpendicular to OT , in P . From P draw the other tangent touching the circle in T' , and cutting OP' , perpendicular to OT' , in P' . Let PC, PM' cut T, T' in S', R ; and let MS , perpendicular to MO , cut PM' in S . Then prove that $P'R, PS'$ are parallel lines, and P', S, S' collinear points.

5552. (T. Mitcheson, B.A., L.C.P.)— $ABCD$ is a quadrilateral in a circle; the sides BA, CD are produced to meet at P ; AD, BC are produced to meet at Q ; PO, QO are drawn bisecting the angles BPC, BQA : find the relation between the lines PO, QO and the tangents from P, Q .

5554. (E. W. Symonds, M.A.)—Prove that the square of the tangent from any point P in the plane of a triangle to the circumscribed circle $= xyz(x \sin \alpha + y \sin \beta + z \sin \gamma) \div 2\Delta$, where x, y, z are the distances of P from the vertices of the triangle, and α, β, γ the angles subtended at P by the sides, so taken that $\alpha + \beta + \gamma = 2\pi$.

5567. (Professor Crofton, F.R.S.)—If z, z' be the two segments into which the chord C divides the area Ω , prove that

$$\iint C^4 dp d\omega = 6 \iint z z' dp d\omega.$$

(See Williamson's *Integral Calculus*, p. 348.)

5568. (Professor Wolstenholme, M.A.)—If the conic of closest contact be described at each point P of a given curve, and C be its centre, (1) prove (as may be done in two lines by taking two consecutive conics) that PC is the tangent at C to the locus of C . If the intrinsic equation of the given curve be $ds/d\theta = c(\sec m\theta)^{2/m}$, and CD be the semi-diameter of the conic conjugate to CP , prove (2) that the ratio $CD^2 : CP^2$ is constant, and, if $m = 3$, both CP and CD are constant.

5577. (A. Martin, M.A.)—Three equal circles touch each other externally; and a point is taken at random in each circle. Find the chance that the triangle formed by joining them is acute.

5583. (J. J. Walker, M.A.)—Express

$4\{(bc-f^2)\sin^2 A + (ca-g^2)\sin^2 B + (ab-h^2)\sin^2 C$
 $+ 2(gh-af)\sin B \sin C + \dots\} - (a+b+c-2f\cos A-2g\cos B-2h\cos C)^2$
 as the sum of two squares. What is the geometrical significance of the transformation?

5584. (Hugh McColl, B.A.)—A confectioner made an immense Christmas cake, in which he put m small gold rings, and invited the public to purchase it, each purchaser to buy as much or as little as he pleased. The cake was bought up in random unequal portions, by n persons. Find the chance that Mr. Jones, who was one of these persons, got a ring (one at least), the cake being supposed to be so large that the rings may be regarded as mere points in comparison. [This question is connected in principle with Questions 5291 and 5318, and, like these, may be solved without the aid of the Integral Calculus.]

5589. (Rev. W. A. Whitworth, M.A.)—Two persons are known to have passed over the same route in opposite directions within a period of time $m+n+r$, the one occupying time m , and the other time n . Find the chance that they will have met.

5595. (Professor Crofton, F.R.S.)—In any convex figure that has a centre O , let a diameter POQ revolve round O , and trace the closed curve that is the locus of the centroid of either half into which POQ divides the area; let A = area of this closed curve. Prove that the mean area of the triangle OXY , formed by the centre O and two points X, Y taken at random in the given convex area, is equal to $\frac{1}{4}A$.

[In the Solution of Question 5307, on p. 20 of Vol. xxviii. of the *Reprint*, it has been already shown that the mean distance of O from a point taken at random in the given area A is equal to one-fourth of the perimeter of A .]

5596. (Professor Asaph Hall, M.A.)—Given

$$x = a \cos A \pm r_1, \quad y = a \sin A \pm r_2,$$

where r_1, r_2 are the probable errors of x, y , find the probable errors of a, A .

5597. (Professor Minchin, M.A.)—A heavy body movable in a vertical plane about a fixed horizontal axis is sustained in a given position by a rope (whose weight may be neglected); state the cases (depending on the natures of the rope and axis) in which the resultant action at the axis can be found both in magnitude and in direction.

5603. (C. Leudesdorf, M.A.)—Within a given circle a point is taken at random, and with this as centre and any radius taken at random a circle is described so as to lie wholly within the first. If the first circle be now reciprocated with regard to the second, find the average area of the resulting ellipse.

5604. (T. Cotterill, M.A.)—A curve U (order m and class n) and a point Z not at infinity are given on a plane. Take the pedal curve of a point of U to the point Z and the antipedal of a tangent to U to the same point, and then find the order of the pedal if the curve U touches the circular asymptotes of Z and the line at infinity Σ times and the class of the antipedal of U if U passes S times through the origin and the circular points at infinity. A similar construction holds for oblique pedals.

5616. (L. H. Rosenthal, B.A.)—If P, Q, p, q are four points on a

parabola such that Pp and Qq are respectively parallel to the tangents at q and p , prove that (1) $PQ = 3pq$; (2) the perpendicular on PQ from the intersection of tangents at p and q is equal to the perpendicular on pq from the intersection of tangents at P and Q ; (3) the perpendicular on PQ from the intersection of the tangents at P and Q is nine times the corresponding perpendicular on pq ; (4) the perpendicular from the intersection of tangents at p and q on PQ is five times that from the same point on pq ; also (5) find the locus of the intersections with PQ of tangents at p and q , (a) when pq passes through focus, and (8) when pq is parallel to a fixed line.

5629. (Professor Crofton, F.R.S.)—Prove that the mean distance of the vertex of a triangle from all points in the area is equal to its distance from the centroid, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centroid in a direction parallel to the other, the axis of the parabola being parallel to the base.

5637. (W. H. H. Hudson, M.A.)—A line PQN is drawn perpendicular to ON , the tangent at O to a curve at a point where the circle of curvature has five-pointic intersection with the curve, cutting the curve and circle of curvature at O in PQ ; prove that PQ varies ultimately as ON^6 , and that $\frac{PQ}{ON^6} = \frac{1}{120\rho^2} \frac{d^5\rho}{ds^5}$ ultimately, where ρ is the radius of curvature at O , and s the arc of the curve measured from a fixed point up to O .

5639. (Col. A. R. Clarke, C.B., F.R.S.)— A, B are two points on the surface of an ellipsoid of small eccentricity; AMB is the curve of intersection with this surface of the plane that contains the normal at A and the point B ; M is the point in which the curve cuts at right angles a meridian line on the ellipsoid; and the angle between the normals at A and B is supposed to be small (say not exceeding 20° or so). Prove that under certain circumstances the geodesic joining A and B will intersect the plane curve AB *between* the points A and B , but not generally; the condition of intersection is that M must be between A and B , and AM must be between $\frac{1}{4}AB$ and $\frac{3}{4}AB$.

5645. (H. MacColl, B.A.)—If from the centre C of a square straight lines $CP_1, CP_2, CP_3, \&c.$, are drawn at equal angular intervals to meet the perimeter in the points $P_1, P_2, P_3, \&c.$, show (1) that (the number of these lines being increased without limit) the average area of the circle of which the variable CP is the radius is equal to the area of the given square; and (2) that this is also true if we substitute equilateral and equiangular polygon for the word square.

5655. (D. Edwardes.)—If a triangle is inscribed in a parabola, its centroid coinciding with the focus, prove that (1) each side touches another parabola, the axes of the two being coincident, and the latus rectum of the latter equal to four times that of the former; (2) the focus of the original parabola is the pole of their chord of intersection, and its vertex lies on the circle circumscribing the triangle.

5656. (J. Dawson.)—Find the locus of a point such that the product of the lines joining it to two given points shall have a given ratio to the sum of the squares of its distances from two other fixed points, and show the peculiarity arising when the given points are on a circle.

5659. (Professor Sylvester, F.R.S.)—Show (1) that the number of distinct chemical compounds that can be formed with three X atoms of valence m and one Y atom of valence n is the coefficient of $a^m x^n$ in the expansion of

$$\frac{(1 + a^2 x^2)(1 + a^3 x^3 + a^4 x^4)}{(1 - a^2)(1 - ax)(1 - a^4 x^2)(1 - a^4 x^4)};$$

and that consequently (2) the number of distinct chemical compounds that can be formed by three m -valent atoms combined with hydrogen is

$$\text{the co-efficient of } x^m \text{ in } \frac{1 - a + a^2 + a^3}{(1 - a)^2(1 - a^2)(1 - a^4)}.$$

[Professor Sylvester states that by "chemical compound" in the above question is to be understood any combination of the atoms which satisfies the law of atomicity without regard to its being a single body or two.]

5660. (Professor Crofton, F.R.S.)—Prove the following relations between mean values relating to ρ , the distance of two points taken at random in any plane convex area, and to C , the length of a chord drawn at random across the same area:—

$$M\left(\frac{1}{\rho}\right) = 3 \frac{M(C^2)}{M(C^3)}, \quad M(\rho) = \frac{1}{2} \frac{M(C^4)}{M(C^3)}.$$

5665. (Professor Casey, M.A.)—Given the distance between the centres of two circles, the radius of one circle, and the length of an endless band that passes round the two circles, find the radius of the other circle.

5666. (Editor.)—Given three circles, each touching the other two, find (1) the area of the triangle that joins their centres, and the radius of the circle that (2) touches all three circles, (3) passes through their points of contact, (4) passes through their centres; also (5) show what the foregoing results become when the three centres are in the same straight line. [Adapted from the *Journal de Mathématiques Élémentaires*.]

5671. (W. S. M'Cay, M.A.)—There are sixteen circles that touch the sides of the four triangles of a quadrilateral. Prove that their centres lie in fours on four coaxial circles.

5681. (C. H. Hinton, M.A.)—Prove (1) that the locus of points from which an ellipse is seen as a circle is an hyperbola passing through the foci of the ellipse; and (2) state and prove the corresponding property of an ellipsoid.

5683. (D. Edwardes.)—Prove that the inscribed circle of a triangle will (1) pass through the orthocentre if

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) = 2 \cos A \cos B \cos C,$$

and (2) through the centroid if

$$3(1 + \cos A)(1 + \cos B)(1 + \cos C) = 8 \cos A \cos B \cos C + 8.$$

5685. (C. Leudesdorf, M.A.)—Solve the equation

$$\left\{x - \frac{1}{2}a(1 - 9a^4)\right\}^2 + \left\{x^3 - \frac{1 + 15a^4}{6a}\right\}^2 = \frac{(1 + 9a^4)^3}{36a^2},$$

and show that it has four equal roots when $a = \pm(45)^{-\frac{1}{4}}$.

5687. (Professor Mannheim.)—On prend un point O d'où l'on voit sous un angle droit la distance qui sépare les foyers d'un pinceau donné. Autour de la parallèle menée de O à un rayon de ce pinceau, on fait

tourner d'un angle droit le plan mené par O et par ce rayon. De la même manière, on fait tourner respectivement dans le même sens, chacun des rayons du pinceau. Après leurs mouvements ces rayons forment un nouveau pinceau : on demande de démontrer que du point O, on voit aussi sous un angle droit la distance qui sépare les foyers de ce pinceau.

5693. (Professor Minchin, M.A.)—Any system of forces acting on a rigid body can be astatically equilibrated by a single force at the centre of the plane of centres, together with two couples in this plane. If the forces of these couples are taken at right angles to each other, their arms (not their *lever arms*) will be in the directions of a pair of conjugate diameters of an ellipse fixed in the body; and, if these arms are taken equal in length to the semi-conjugate diameters of this ellipse, or any one similar, concentric, and similarly placed, the forces of the two couples will be all equal, and constant for the same ellipse.

5697. (Editor.)—A point is taken at random within a circle, and a chord is drawn at right angles to the straight line joining the point with the centre. Find the average of the ratio of the parts into which the chord divides (1) the circumference, and (2) the circle; also (3) the probability that either ratio will be less than a given magnitude.

5702. (Rev. W. A. Whitworth.)—A. and B. play at a game which cannot be drawn, and the odds are $\mu : \nu$ in favour of A. winning any assigned game. They agree to play until either A. shall have won by scoring m games ahead of B., or B. shall have won by scoring n games ahead of A. Show that, when A. has scored p games more than B., the odds in favour of his winning the match are

$$\mu^{m+n} - \mu^{m-p} \nu^{n+p} : \mu^{m-p} \nu^{n+p} - \nu^{m+n}.$$

5703. (Rev. W. G. Wright, Ph.D.)—A chord is drawn through the point (2, 0) in the ellipse whose semi-axes are 3 and 2. Find the locus of the intersection of normals from the ends of this chord.

5704. (W. H. H. Hudson, M.A.)—In a subtraction sum in the scale of which the radix is r , the numbers being selected at random, the odds against having to carry one after the p th subtraction to the $p+1$ th figure of the subtrahend are $rp+1 : rp-1$.

5707. (R. A. Roberts, M.A.)—Prove that any tangent to either of the conics $x^2 + y^2 + (1-3k^2)x^2 \pm 2k\sqrt{3}xy = 0$ is divided in a constant anharmonic ratio by the quartic $y^2x^2 + y^2x^2 + x^2y^2 - k^2x^4 = 0$.

5709. (A. Martin.)—Three random points being taken inside a sphere as the corners of a triangle, find the chance that all the sides of the triangle are less than the radius of the sphere.

5711. (C. Leudesdorf, M.A.)—The $(a+1)$ th, $(b+1)$ th, $(c+1)$ th terms of a certain arithmetical series are equal to the $(b+1)$ th, $(c+1)$ th, $(a+1)$ th terms respectively of a certain harmonical series, and their first terms are the reciprocals one of the other. Determine the series.

5715. (Professor Wolstenholme, M.A.)—Prove that the general equation of conics confocal with $u = 0$ is

$$\mu^2 \Delta + \mu \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} \right) \left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\} = 2(\mu - 8) \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} \right)^2 u,$$

where Δ is the discriminant of u . If $\mu = 16$, this is the confocal whose asymptotes lie along the equi-conjugate diameters of u ; and, if $\mu = 8$, it is the confocal round which can be described parallelograms inscribed in u . Another form of the former confocal is

$$\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right)\left\{\frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2 + \frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2 - 2\frac{d^2u}{dx dy}\frac{du}{dx}\frac{du}{dy}\right\} \\ = 2\left\{\frac{d^2u}{dx^2}\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx dy}\right)^2\right\}\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right\}.$$

5716. (Professor Ball, F.R.S.)—The image of a star is formed by an achromatic object glass. Very near the image, and between it and the object glass, a plano-cylindrical lens is placed, whose focal length is considerably larger than its distance from the image. The image was a point. What form has it now? [This arrangement occurs in the application of the spectroscope to astronomical inquiries.]

5717. (J. W. L. Glaisher, F.R.S.)—An infinite number of perfectly flexible strings of the same length are knotted together at one end, and thrown down at random. Find the density at any distance from the knot.

5720. (Editor.)—Through a point taken at random inside a circle, a random chord is drawn. Find (1) the probability that a second chord so drawn at random will intersect the first inside the circle; and (2), if a third such random chord is drawn, find the respective probabilities that the three chords will form 0, 1, 2, 3 intersections inside the circle.

5722. (Christine Ladd.)—Prove that the Pascals of Question 5670 are eighteen in number, and that they pass by threes through the six points h which lie by threes on two g lines which are harmonic conjugates with respect to the conic.

[The notation is that of the note at the end of the fifth edition of Salmon's *Conics*.]

5726. (H. W. Harris, B.A.)—Let the normal at a point P on an ellipse meet the curve again at Q. From Q draw a parallel to the diameter at P, meeting the tangent at P in a point O. Find the locus of the point O.

5730. (S. Tebay, B.A.)—In a game of n points, every time the player is successful he scores one point, but, when he is unsuccessful, he forfeits all the points he has gained, and then proceeds to the end of the game. If p be the measure of his skill, required (1) the probability that he wins with at least m points; (2) if the game terminates when the player has scored m points, required his probability in this case; and (3) apply these considerations to the cases in which the acquisition of a point does not depend upon skill, but only on promiscuous arrangement. [See Question 5564.]

5731. (R. A. Roberts, M.A.)—If from

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} - 4 = 0, \quad (2) a^2x^2 + b^2y^2 - (a^2 + b^2)^2 = 0,$$

tangents be drawn to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv S = 0$, prove that they form, with their chord of contact, a triangle whose (1) centre of gravity, (2) intersection of perpendiculars, lies on $S = 0$.

5734. (A. Martin.)—A. has m dollars and B. n dollars, and they play at a game for which it is b to a that B. shall win any one game. Prove that the chances for the ultimate success of each player are respectively

$$\frac{a^m (b^m - a^m)}{b^{m+n} - a^{m+n}} \quad \text{and} \quad \frac{b^n (b^n - a^n)}{b^{m+n} - a^{m+n}}.$$

5741. (Editor.)—A conic is drawn on a right cone. Find the equation of the curve when the surface of the cone is developed or spread out on a tangent plane.

5747. (Rev. Joseph A. Galbraith, M.A.)—Let G, g_1, g_2, g_3 be the centroids of a triangle and of the equilateral triangles drawn on its sides externally. Draw circles on the lines Gg_1, Gg_2, Gg_3 as diameters; draw diameters of these circles passing through the middles of the sides of the triangle; and write m_1, m_2, m_3 at the extremities of the greater segments, and n_1, n_2, n_3 at the extremities of the lesser segments of these diameters. Then prove that (1) the points m_1, m_2, m_3 , also the points n_1, n_2, n_3 , are in a straight line, the lines containing them intersecting at right angles at the point G ; (2) the diameters so drawn, if produced, meet in a point, and make angles of 60° with each other; (3) the lines m_1, m_2, m_3 and n_1, n_2, n_3 are the principal axes of the triangle; (4) the principal moments of inertia of the area of the triangle are equal to the area multiplied by half of the squares of the greater and lesser segments of the diameters.

5756. (D. Edwardes.)—If $(1+x+x^2)^n = A_0 + A_1x + A_2x^2 + \&c.$, prove that $A_{n+2} + nA_{n+1} + \frac{n(n-1)}{2!}A_n + \dots + nA_3 + A_2 = (-)^n A_{n+2}.$

5757. (R. E. Riley, B.A.)—In Todhunter's figure for Euclid I. 47, if AD, FC meet at M; AE, BK at N; and BM, CN at O, prove that O is the radical centre of the circles that circumscribe the three squares in the figure.

5759. (W. J. Wright, Ph.D.)—Find in how many ways the number k may be made up of lesser numbers, no one of which falls short of n , counting orders, or permutations, as ways.

5778. (Wilkins.)—If a hyperboloid of one sheet pass through three given lines in space, obtain its centre by a geometrical construction.

5788. (S. A. Renshaw.)—Circles are drawn on three chords of a circle OA, OB, OC as diameters: that on OA cuts the lines OB, OC in X and Y; that on OB cuts OA, OC in H and G; and that on OC cuts OA, OB in I and T. Prove that XY, HG, IT intersect each other in a common point situated on the line PQR, P, Q, R being the other three intersections of the circles described on the chords.

5797. (Professor Minchin, M.A.)—It is known (see *Nature* for June 6th, 1878) that, if one plane be moved about in any manner on another plane which is fixed, and if the first plane be finally brought back to its original position, those points on it which have described curves of equal areas all lie on a circle the position of whose centre is independent of the magnitude of the common area of the curves. Determine whether an analogous result can be found for the displacement of a movable spherical surface inside an equal fixed one.

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